Definition of Fuzzy Inference Systems based on f-inclusion

V Jornadas Científicas DATAI

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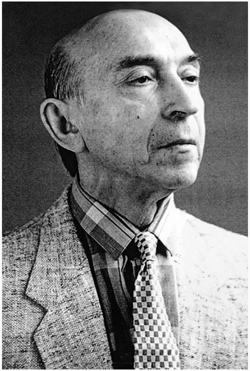
A brief introduction to Fuzzy Logic

What is Fuzzy Logic?

Fuzzy Logic is a branch of mathematics that was first introduced by *Lotfi A. Zadeh* in 1965.



- **Essence:** assertions are not either “true” or “false”: there is a wide range of classifications between them.
- **Objective:** to incorporate reasoning that accommodates vagueness and uncertainty.
- **Mechanism:** propositions are classified via fuzzy sets, which assign to objects a value in $[0, 1]$, where 0 stands for “false” and 1 stands for “true”.

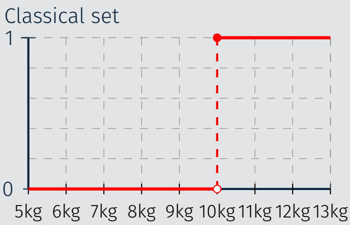
Definition of fuzzy set

Consider the set of elements $\mathcal{U}$ we want to classify, called **universe**. A **fuzzy set** of $\mathcal{U}$ is a mapping

$$\begin{aligned} A\colon \quad \mathcal{U} &\longrightarrow [0,1] \\ u &\longmapsto A(u). \end{aligned}$$

For each $u \in \mathcal{U}$, $A(u)$ is called the **membership degree** of u in A.

Example. Fuzzy set A="Heavy laundry load"



Operations in Fuzzy Logic

Operators in fuzzy logic...

- Are functions used to combine multiple fuzzy sets.
- Translate the **connectives of natural language** into a mathematical framework.
- Are used to model **inference rules**.

Name	Expression	Symbol	Formula
Conjunction	AND	$A \wedge B$	$\min\{A(x), B(x)\}$
Disjunction	OR	$A \vee B$	$\max\{A(x), B(x)\}$
Implication	IF...THEN...	$A \rightarrow B$	$\max\{1 - A(x), B(x)\}$
Negation	NOT	$\neg A$	$1 - A(x)$

Inference Rules in Fuzzy Logic

Generalized Modus Ponens

Modus Ponens **(MP)**

$$\begin{array}{llcl} & x \text{ is } A & \rightarrow & x \text{ is } B \\ & t \text{ is } A & & \\ \hline \therefore & & & t \text{ is } B \end{array}$$

Example: Modus Ponens

$$\begin{array}{ll} & \mathit{IF} \text{ laundry load is } \geq 10\text{kg}, \ \mathit{THEN} \text{ wash cycle is } \textbf{long} \\ & \text{Laundry load is 11kg} \\ \hline \therefore & \text{Wash cycle is } \textbf{long} \end{array}$$

Generalized Modus Ponens

Modus Ponens **(MP)**

$$\begin{array}{llcl} & x \text{ is } A & \rightarrow & x \text{ is } B \\ & t \text{ is } A & & \\ \hline \therefore & & & t \text{ is } B \end{array}$$

Generalized Modus Ponens **(GMP)**

$$\begin{array}{llcl} & x \text{ is } A & \rightarrow & x \text{ is } B \\ & t \text{ is } A' & & \\ \hline \therefore & & & t \text{ is } B' \end{array}$$

Example: Generalized Modus Ponens

$$\begin{array}{ll} & \mathit{IF} \text{ laundry load is } \geq 10\text{kg}, \;\; \mathit{THEN} \text{ wash cycle is long} \\ & \text{Laundry load is } 9.5\text{kg} \\ \hline \therefore & \text{Wash cycle is } \textbf{medium-long} \end{array}$$

How can we model GMP?

To model a Generalized Modus Ponens, we need:

- A fuzzy **implication operator** to model the relation

$$x \text{ is } A \rightarrow x \text{ is } B.$$

- A **similarity function** that allows us to measure the relationship between A' and A.

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To model a Generalized Modus Ponens, we need:

- A fuzzy **implication operator** to model the relation

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Which operators should we use?

There exists an operator that can fulfill both roles simultaneously, and it is called...

f-inclusion between fuzzy sets

f-inclusion between fuzzy sets

How do we measure inclusion in logic?

In classical logic, given two sets X and Y, X is included in Y if every element that belongs to X also belongs to Y. Formally stated...

$$X \subseteq Y \text{ if and only if } \forall u(X(u) \to Y(u))$$

Degree of inclusion between fuzzy sets

Until now, the standard inclusion between two fuzzy sets A and B has been defined as a value in the interval $[0, 1]$ obtained from an **implication operator**.

$$S_{\to}(A, B) = \bigwedge_{u \in \mathcal{U}} (A(u) \to B(u))$$

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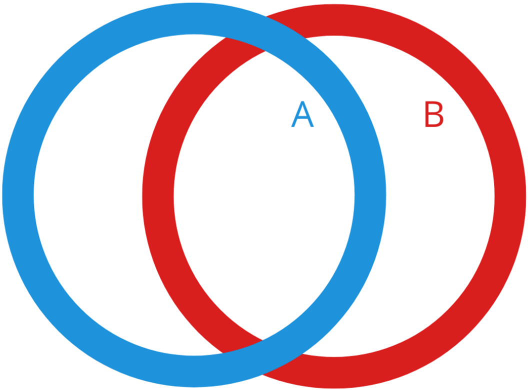
What if we create an implication operator from an inclusion measure?

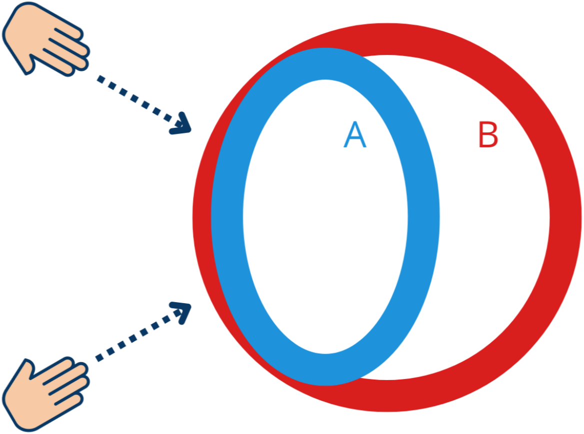
Definition of f-inclusion

An intuitive idea...



Definition of f-inclusion

An intuitive idea...



f-inclusion between fuzzy sets

What functions can we use?

We define the set of indexes of inclusion Ω as the set of functions $f\colon [0,1] \to [0,1]$ satisfying for all $a, b \in [0,1]$:

- **Increasing monotonicity:** if $a \leq b$, then $f(a) \leq f(b)$.
- **Deflation:** $f(a) \leq a$.

f-index of inclusion between fuzzy sets

Given two fuzzy sets A and B defined on a universe $\mathcal{U}$ and an index of inclusion $f \in \Omega$, A is said to be **f-included** in B ($A \subseteq_f B$) if:

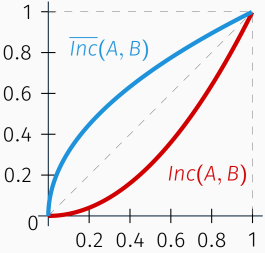
$$f(A(u)) \leq B(u) \text{ for all } u \in \mathcal{U}.$$

The f-index of inclusion is the greatest function satisfying $A \subseteq_f B$.

GMP based on f-inclusion

We denote the f-index of inclusion of A in B by $Inc(A, B)$. We can model a Generalized Modus Ponens:

- taking the **adjoint** ($\overline{Inc}(A, B)$) of the f-indexes of inclusion involved.
- Computing their **composition**.

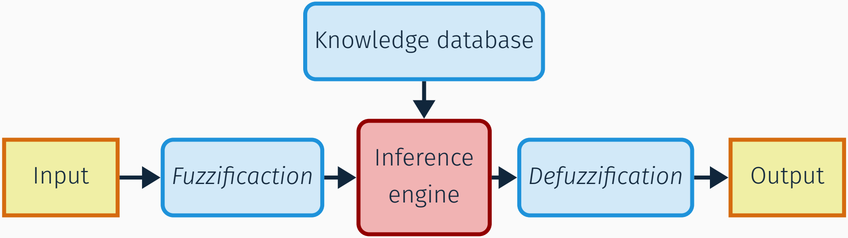


Modelization of GMP

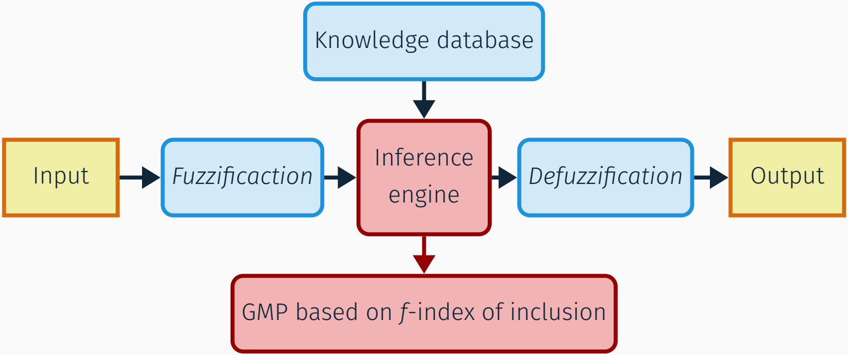
$$
\begin{array}{llllll}
 & \text{IF} & x \text{ is } A & \text{THEN} & y \text{ is } B & \equiv Inc(A, B) \\
 & & x \text{ is } A' & & & \equiv Inc(A', A) \\
\hline
\therefore & & & & y \text{ is } B' & \equiv \overline{Inc}(A', A) \circ \overline{Inc}(A, B)(A')
\end{array}
$$

Fuzzy Inference Systems Based on f-inclusion

Structure of a Fuzzy Inference System



Structure of a Fuzzy Inference System

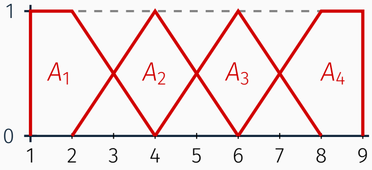


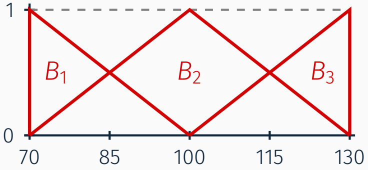
Frame of the FIS

Example. Assume that we conduct a study on heart rate in infants and children between the ages of 1 and 9. We have to build a **frame** consisting of the two universes of study, which are:

- the interval $X = [1, 9]$ representing the ages between 1 and 9.
- the interval $Y = [70, 130]$ which covers a wide range of heart rates (in bpm).

For each universe, we build a fuzzy partitions which distinguish 4 different age ranges in X and 3 levels of heart rate in Y.





Constructing the Knowledge Database (KB)

Our **KB** is composed of **rules** in the form

$$\langle A_i \rightarrow B_j \quad ; f_{ij} \rangle,$$

where:

- A_i is an fuzzy set in the partition in X;
- B_j is an fuzzy set in the partition in Y;
- $f_{ij} \in \Omega$ is an index of inclusion.

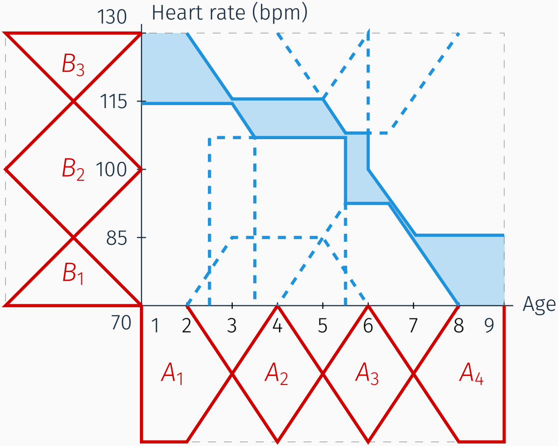
	B_1	B_2	B_3
A_1	f_{11}	f_{12}	f_{13}
A_2	f_{21}	f_{22}	f_{23}
A_3	f_{31}	f_{32}	f_{33}
A_4	f_{41}	f_{42}	f_{43}

How do we interpret these rules?

The index of inclusion that appears in each rule represents the relationship between the two associated fuzzy sets. **The greater the value of the function, the stronger the relationship between them.**

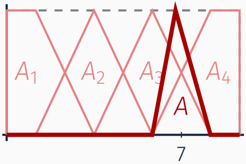
Representation of the Knowledge Database (KB)

When we represent the KB on a plane, we obtain the region (shaded in blue) of data that satisfy relations indicated in the rules.

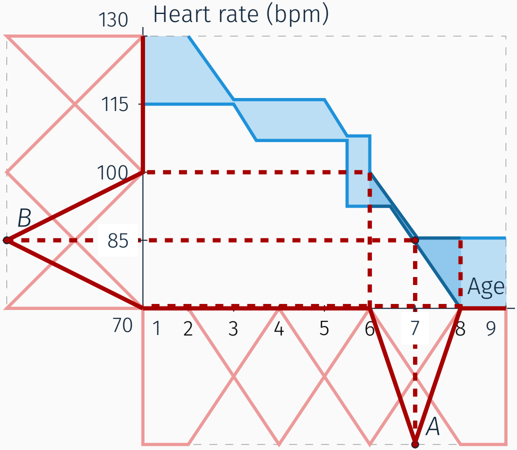


Application of the inference engine

When we take the following input in X:



and apply the inference engine (GMP based on f-inclusion), we obtain an **output** B in the universe Y.



What have we achieved yet?

Desirable properties in our FIS...

We aim to develop a fuzzy inference system...

- With a formal (logical) system structure, including:
  - a syntax;
  - a semantics;
  - a definition **logical consequence**.
- With an applied orientation:
  - a symple theoretical structure;
  - A procedure for constructing knowledge bases from data.

What should our FIS satisfy?

Correctness

A FIS is **correct** if everything that can be proven (inferred) within the system is true.

Completeness

A FIS is **complete** if everything that is true can be proven (inferred) within the system.

What should our FIS satisfy?

Correctness

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Completeness

A FIS is **complete** if everything that is true can be proven (inferred) within the system.

¡We have already proved the correctness of the FIS!

Conclusions and Future Work

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¿What do we want to accomplish?

- We want to obtain results of **approximation of functions** with the desired accuracy.
- We want to study further procedures related to the **FIS**:
  - Fuzzification and defuzzification processes.
  - Construction of knowledge bases from data.
- ¿Would it be possible to obtain a **completeness result**?

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