Closed-form solutions of the jet engine fuel consumption problem during aircraft take-off, climb and cruise

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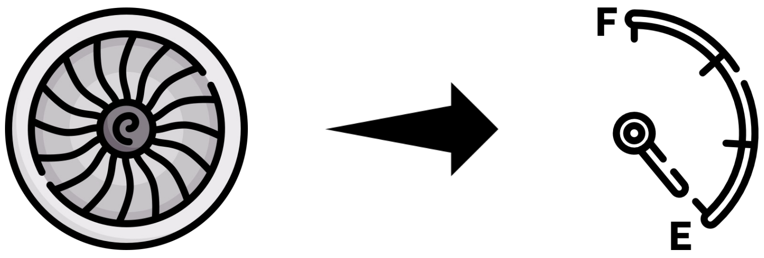
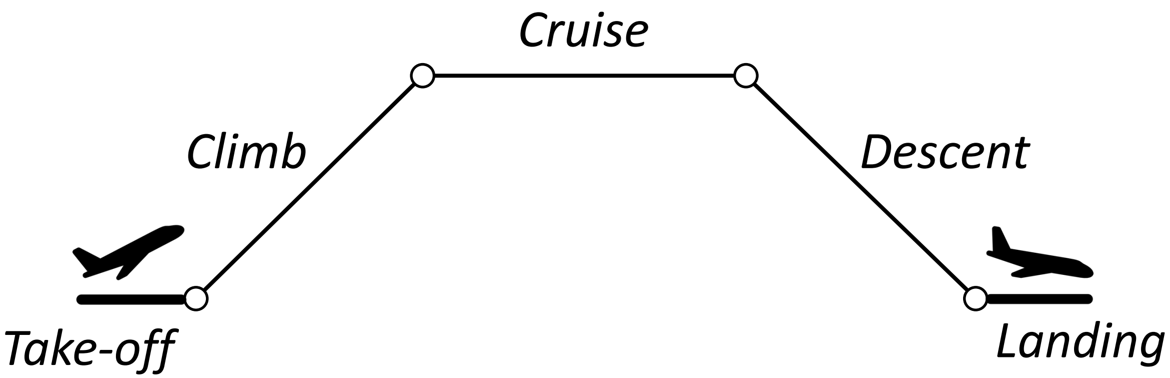
Closed-Form Solution

$$\frac{dy}{dx} = f(x) \rightarrow y(x) = \int f(x) \rightarrow y(x) = F(x) + C$$

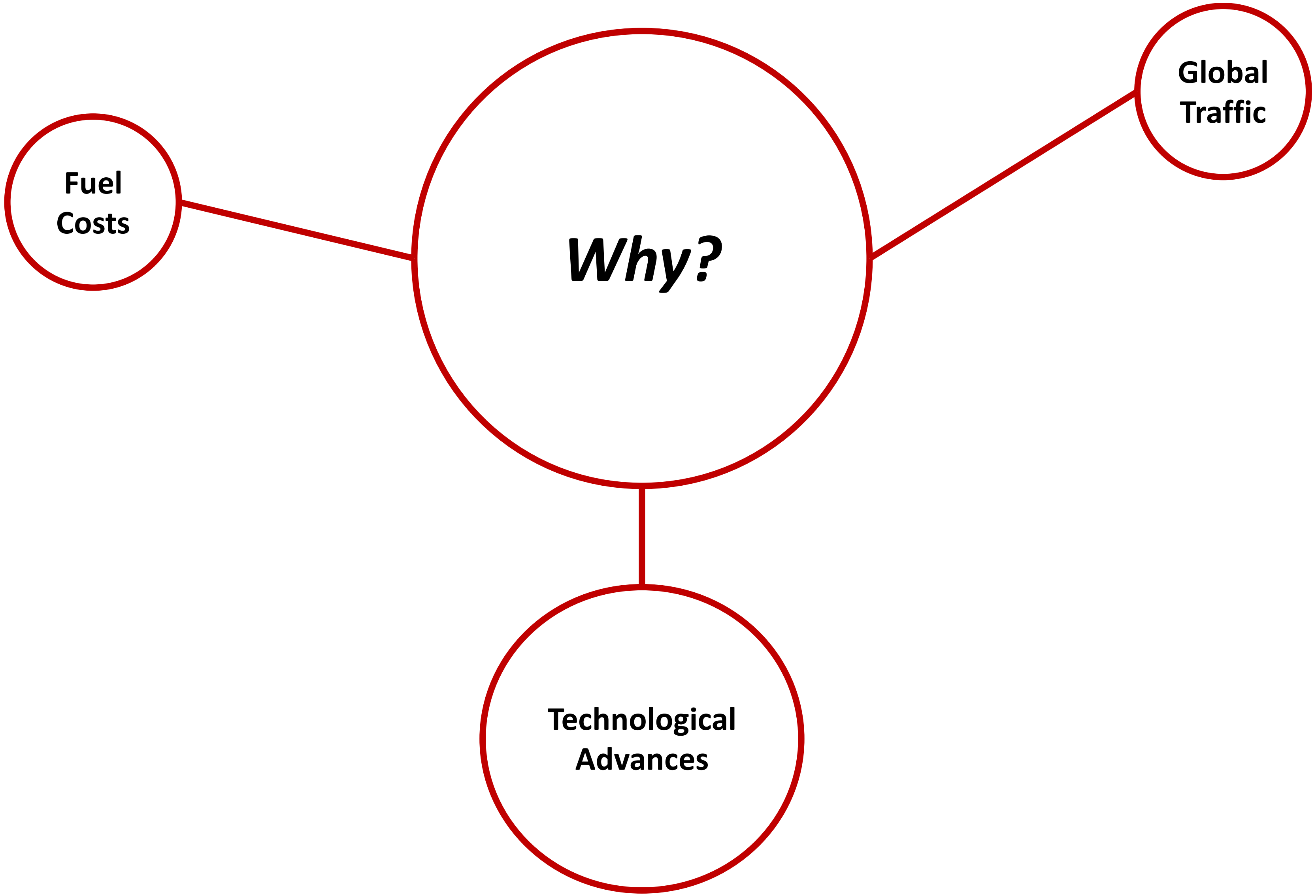
Jet Engine Fuel Consumption Problem



Aircraft Take-off, Climb and Cruise



Why?

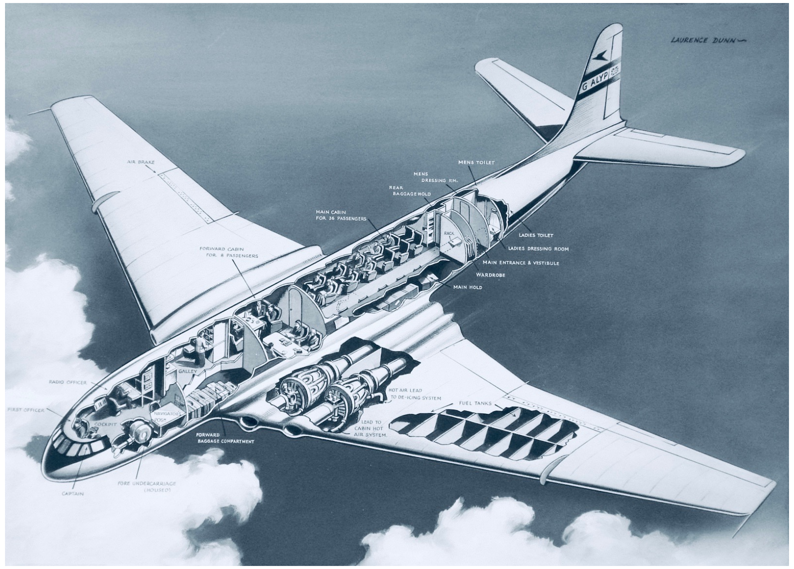

Why?
Fuel Costs
Global Traffic
Technological Advances

Fuel costs

915 million tons
2% human-induced emissions
12% general transport emissions

25% share of airline expenses

Technological Advances



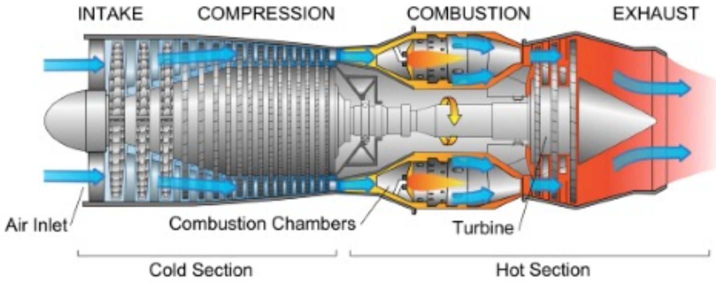


Diagram of a TurboJet

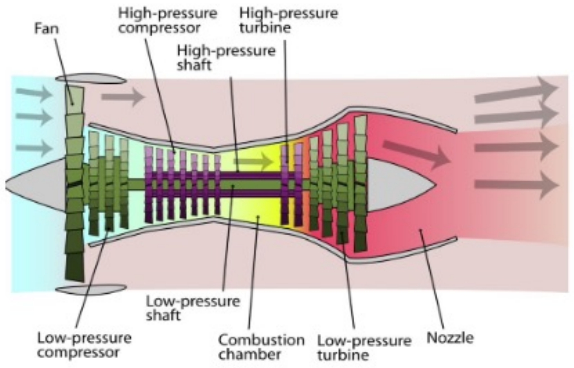


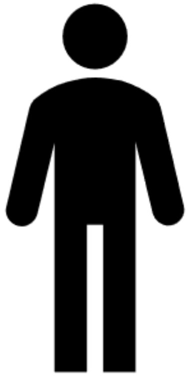
Diagram of a Turbofan

https://www.thisdayinaviation.com/8-april-1954/

https://www.century-of-flight.net/turbojet-vs-turbofan-explained/

https://blog.klm.com/jet-engine-propulsion-the-comparison-of-power-between-a-car-and-an-aircraft/

Global Passenger/Freight Traffic



3 million to 4.4 billion passengers
1970's - 2019

15.5 to 215.1 million tonnes per km
1970's - 2019

Current Literature

Energy Balance Methods

$$E_{F_N} + E_D = Wh + \frac{1}{2}\frac{W}{g}v^2$$

Rutowski, E. S. (1954). "*Energy approach to the general aircraft performance problem*". Journal of the Aeronautical Sciences, *21*(3), 187-195.

Statistical Learning Methods

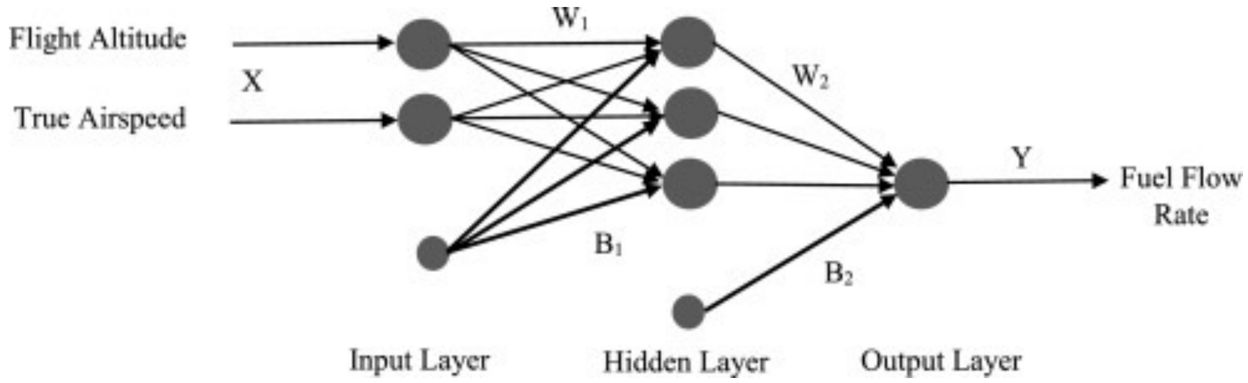


Figure. Topology of a three-layer fuel flow rate FNN.

Baklacioglu, T. (2016). "*Modeling the fuel flow-rate of transport aircraft during flight phases using genetic algorithm-optimized neural networks*". Aerospace Science and Technology, *49*, 52-62.

https://www.redbull.com/es-es/everest-peligros-muertes-avalanchas

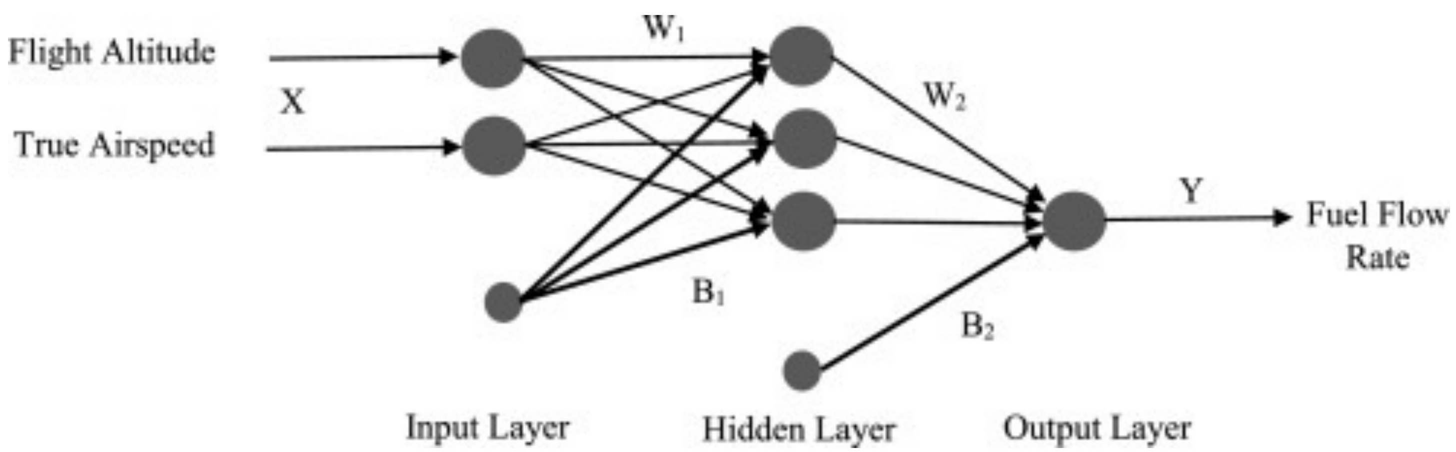
Methodology

Numerical Solution Methods

$$E_{F_N} + E_D = Wh + \frac{1}{2}\frac{W}{g}v^2$$

Closed-Form Solutions

Statistical Learning Methods



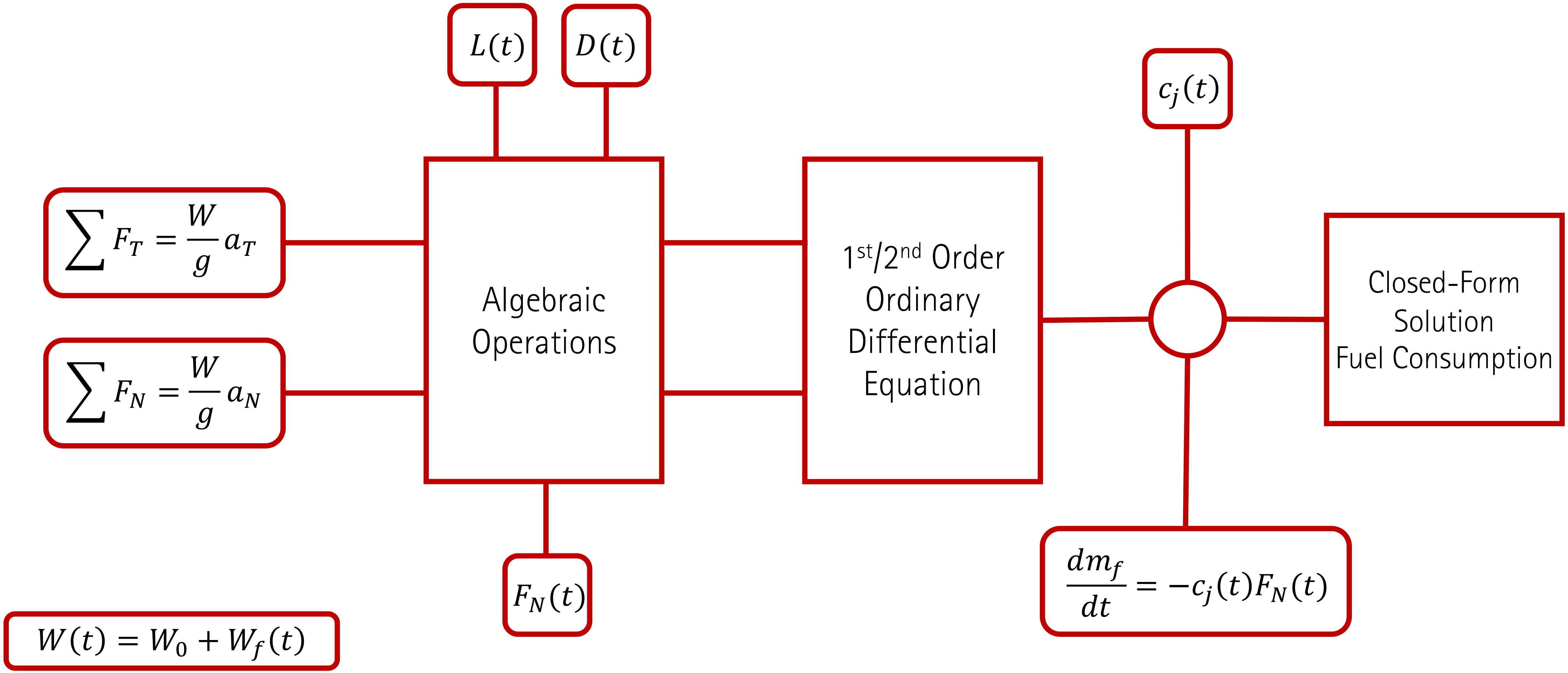
Closed-Form Solutions

Aircraft Performance is a discipline that studies the behavior of an aircraft flying, according to the respective equations of motion of each flight phase.

$$\frac{dm_f}{dt} = c_j(t)F_N(t) \Rightarrow m_f(t) = \int_0^t c_j(\epsilon)F_N(\epsilon)d\epsilon$$

dm_f/dt [kg/s]: aircraft's fuel flow rate
$c_j(t)$ [(kg/s)/N]: thrust specific fuel consumption
$F_N(t)$ [N]: aircraft's thrust

Closed-Form Solutions



Take-off Flight Phase

https://www.planeskies.com/photos/view/5198/airbus/a300-600/n135up

Take-off Flight Phase – Mathematical Model

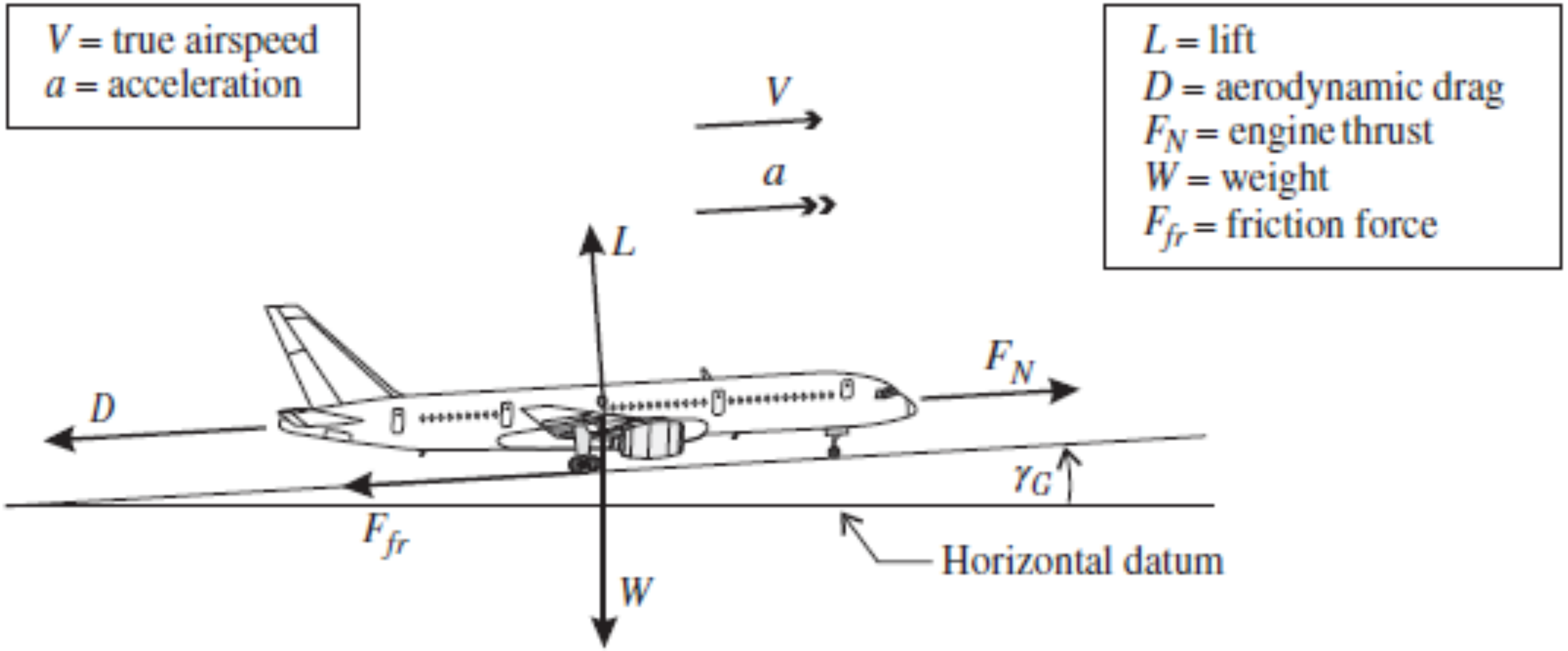


Figure. Force diagram of an aircraft during take-off.

Filippone, A. (2017). "*Advanced Aircraft Flight Performance*," Figure 9.7 p. 234

Equations of Motion

$$F_N(v(t)) - D(v(t)) - F_{fr}(v(t)) = \frac{W(v(t))}{g}\frac{dv}{dt}$$

$$F_{fr}(v(t)) = W(v(t)) - L(v(t))$$

Empirical Relations

$$W(t) = W_0 + W_f(t)$$

$$\frac{dW}{dt} = \frac{dm_f}{dt}g = -c_j(v(t))F_N(v(t))g$$

Aerodynamic Forces

$$L(v(t)) = q(v(t))Ac_L$$

$$D(v(t)) = q(v(t))Ac_D$$

$$q(v(t)) = (1/2)\rho(v(t))^2$$

Empirical Relations

$$c_j(t) = c_{j,0} + c_{j,1}v(t)$$

$$F_N(v(t)) = F_2(v(t))^2 - F_1 v(t) + F_0$$

Equations of Motion

$$F_N(v(t)) - D(v(t)) - F_{fr}(v(t)) = \frac{W(v(t))}{g}\frac{dv}{dt}$$

$$F_{fr}(v(t)) = W(v(t)) - L(v(t))$$

Empirical Relations

$$W(t) = W_0 + W_f(t)$$

$$\frac{dW}{dt} = \frac{dm_f}{dt}g = -c_j(v(t))F_N(v(t))g$$

2nd Order Non-linear Ordinary Differential Equation

$$\frac{d^2v}{dt^2} = \psi(v(t))\left(\frac{dv}{dt}\right)^2 + \phi(v(t))\left(\frac{dv}{dt}\right) + \tau(v(t))$$

1st Order Linear Ordinary Differential Equation

$$\frac{dv}{dt} = a_2 v(t)^2 + a_1 v(t) + a_0$$

Take-off Flight Phase - Results

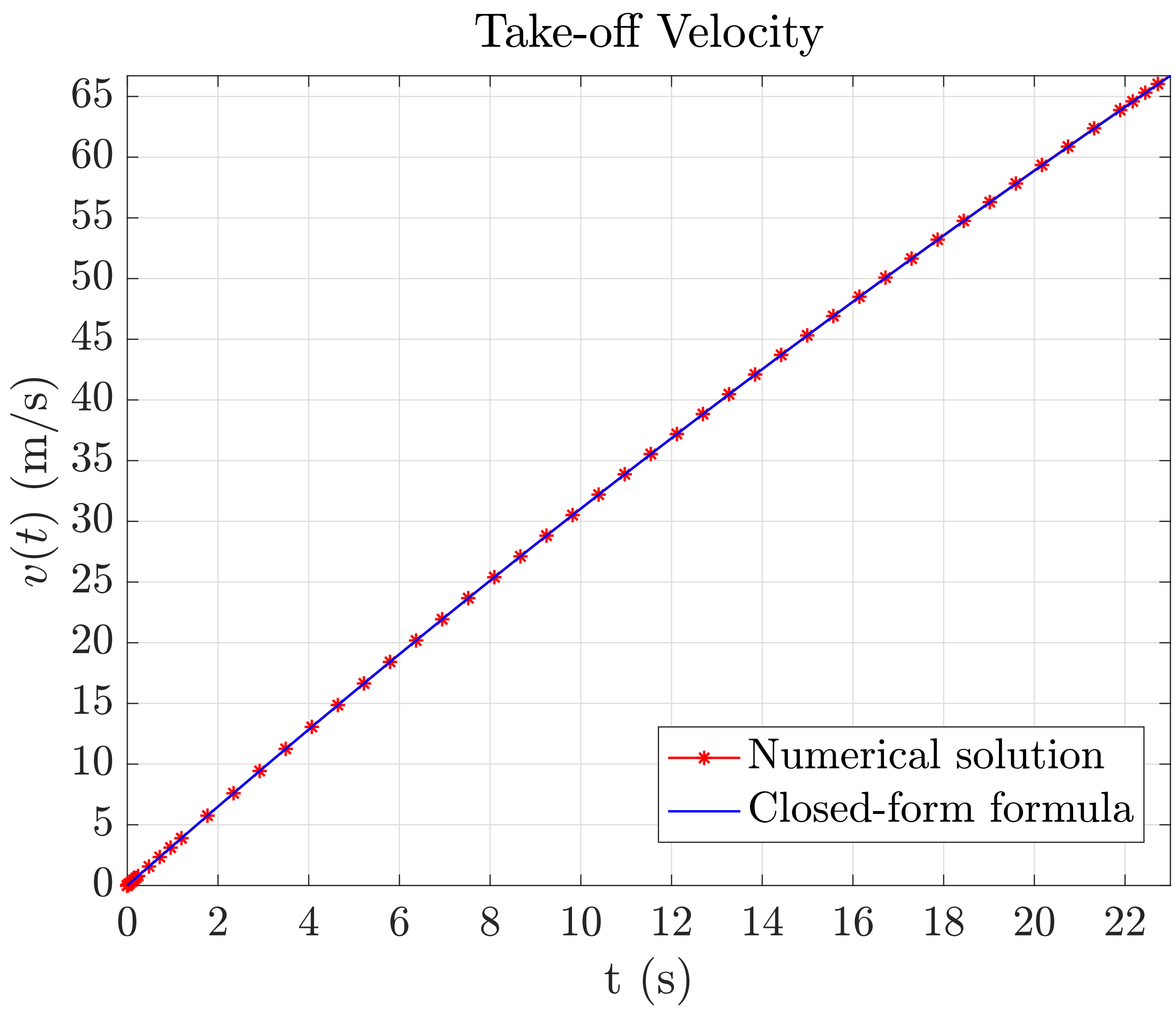


Figure. Velocity variation over time.

Closed-Form Solution of the Aircraft's Velocity

For the initial value problem (IVP) $v(0) = 0$:

$$v(t) = \frac{\sqrt{\Delta}}{2a_2}\left(1 + \frac{a_1^2}{\Delta}\right)\frac{\tan\left(\sqrt{\Delta}t/2\right)}{1 - \left(a_1/\sqrt{\Delta}\right)\tan\left(\sqrt{\Delta}t/2\right)}$$

$$\Delta = 4a_2a_0 - a_1^2 > 0$$

$$v_{to} = v(t_{to}) = 66\ \mathrm{m/s} \sim 237.6\ \mathrm{km/h}$$

$$v_{to} = 1.2v_r \implies v_r = \sqrt{\frac{2W(0)}{\rho_0 A c_{L,max}}}$$

Take-off Flight Phase - Results

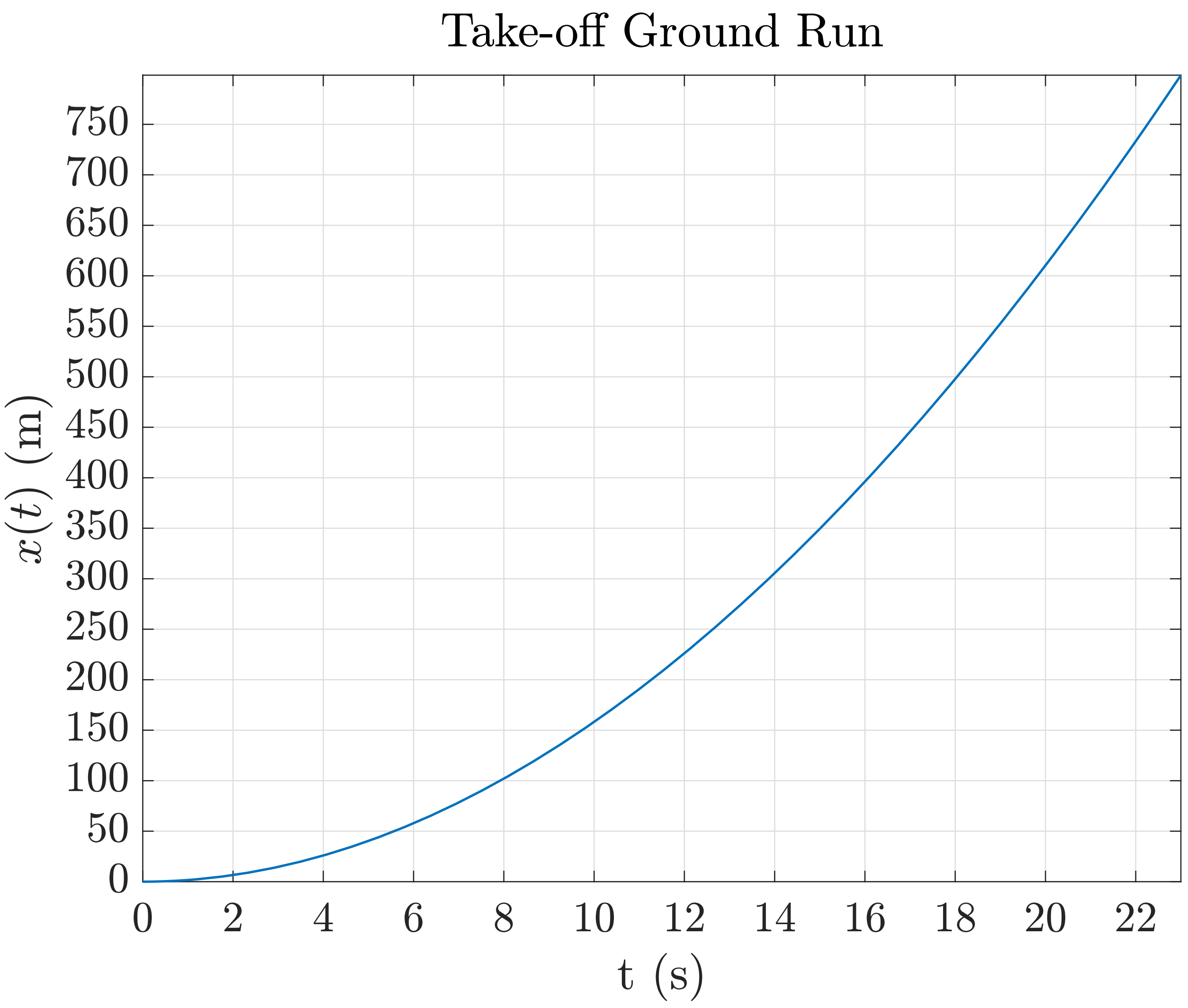


Figure. The aircraft's *ground run* over time.

Closed-Form Solution of the Aircraft's Ground Run

For the initial value problem (IVP) $\mathbf{x(0) = 0}$:

$$x(t) = \int_0^t v(\zeta)d\zeta$$

$$x(t) = -\frac{a_1 t}{2a_2} - \frac{1}{a_2}\ln\left|\cos\left(\frac{\sqrt{\Delta}t}{2}\right) - \frac{a_1}{\sqrt{\Delta}}\sin\left(\frac{\sqrt{\Delta}t}{2}\right)\right|$$

$$x_{to} = x(t_{to}) = 798\text{ m}$$

Take-off Flight Phase - Results

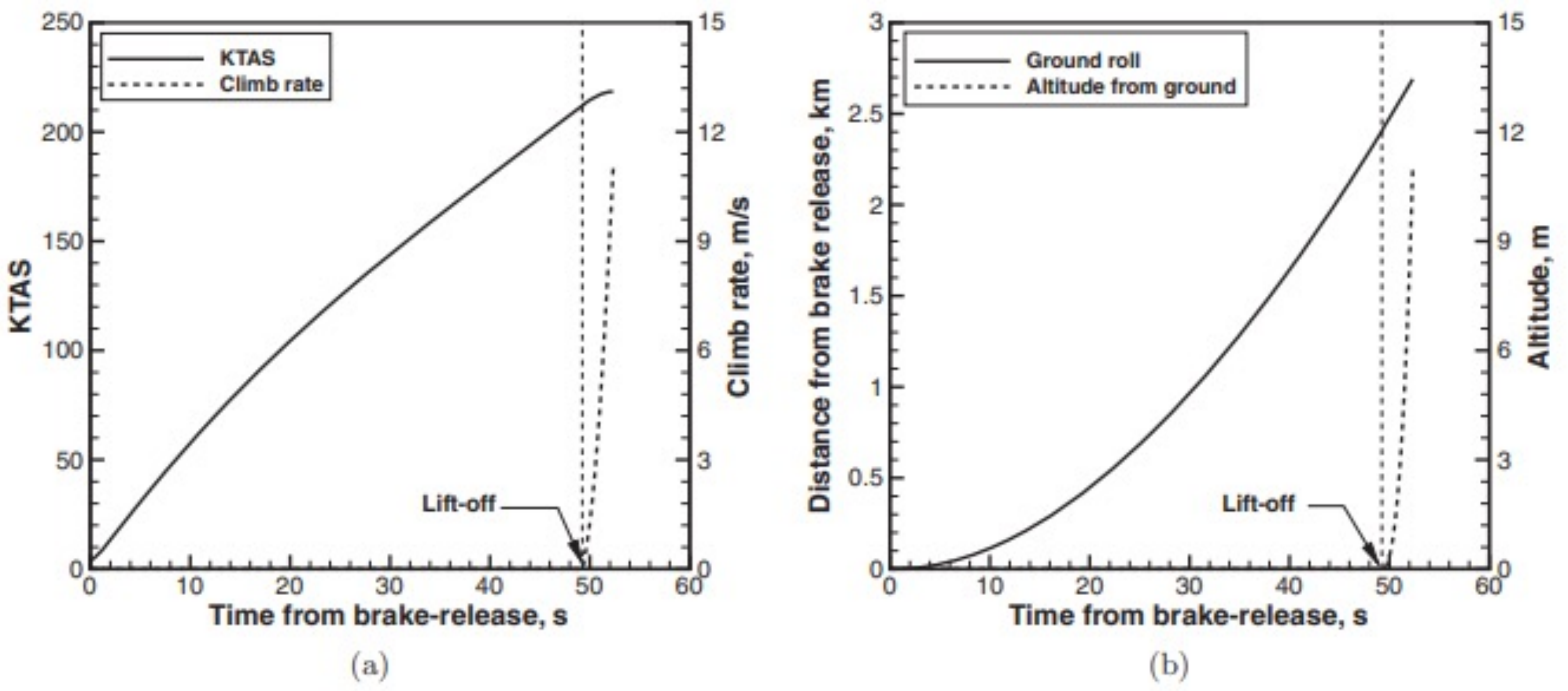


Figure. Velocity and ground run during the take-off flight phase.

Filippone, A. (2017). "*Advanced Aircraft Flight Performance*," Figure 9.7 p. 234

Take-off Flight Phase - Results

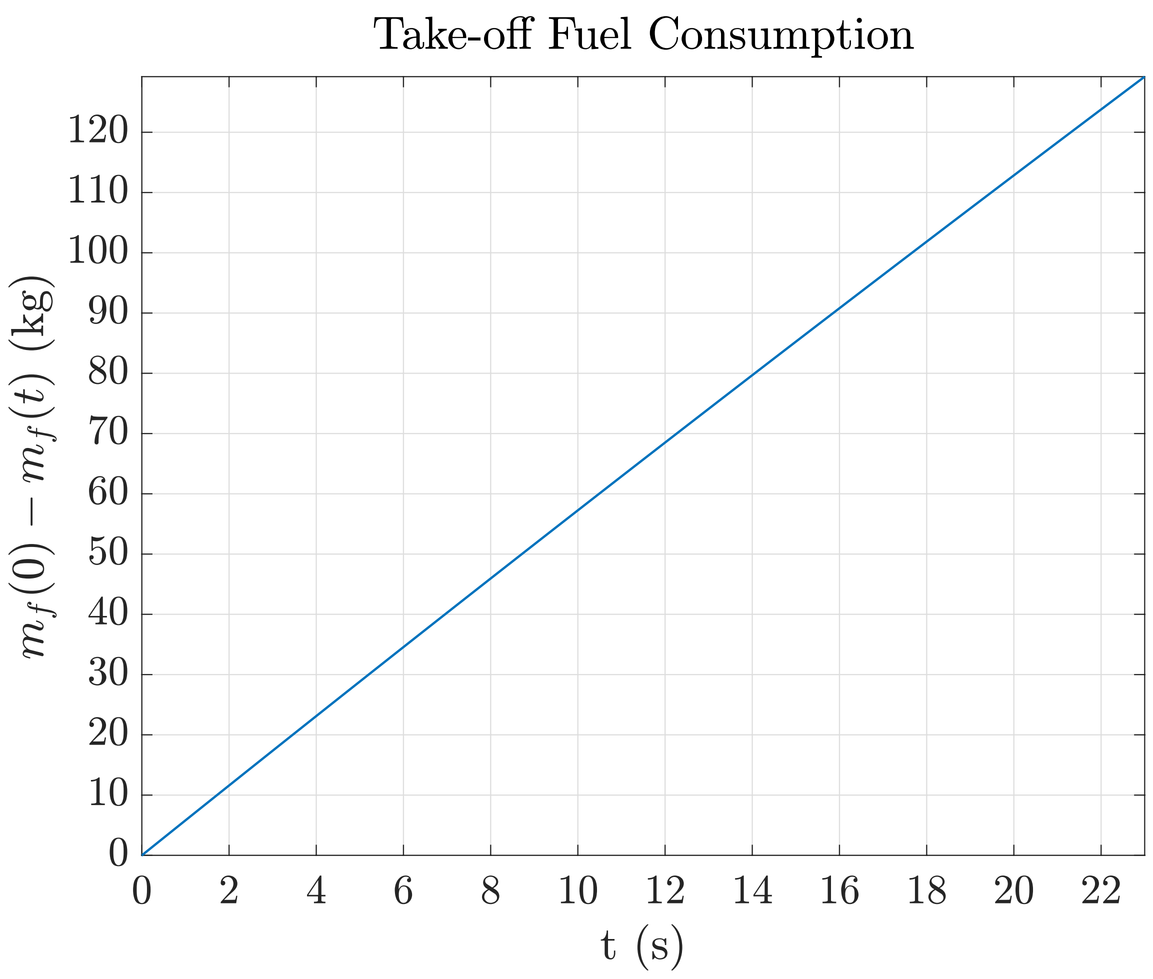


Figure. Fuel consumed during the take-off flight phase.

Closed-Form Solution of the Aircraft's Fuel Consumption

For the initial value problem (IVP) $\mathrm{m_f}(0) = m_{f,0}$:

$$\frac{dm_f}{dt} = c_j\big(v(t)\big)F_N\big(v(t)\big)$$

$$\frac{dm_f}{dt} = \big(c_{j,0} + c_{j,1}v(t)\big)\Big(F_2\big(v(t)\big)^2 - F_1 v(t) + F_0\Big)$$

$$m_{f,to} = m_f(t_{to}) = 129 \text{ kg}$$

$$m_{f,Piano-X} = 111 \text{ kg}$$

Take-off Flight Phase - Results

- Aircraft Performance Model
- Pollutant Gas Emissions:
  - ➢ ICAO Engine Emissions Databank – LTO cycle
  - ➢ Emission Indices (EI) of pollutant gases

$$p_i = \Delta m_f \cdot EI_i \implies p_{CO_2} = \Delta m_f \cdot EI_{CO_2}$$

Cruising Flight Phase

https://www.airliners.net/photo/Qantas/Boeing-747-438/1729381/L

Cruising Flight Phase – Mathematical Model

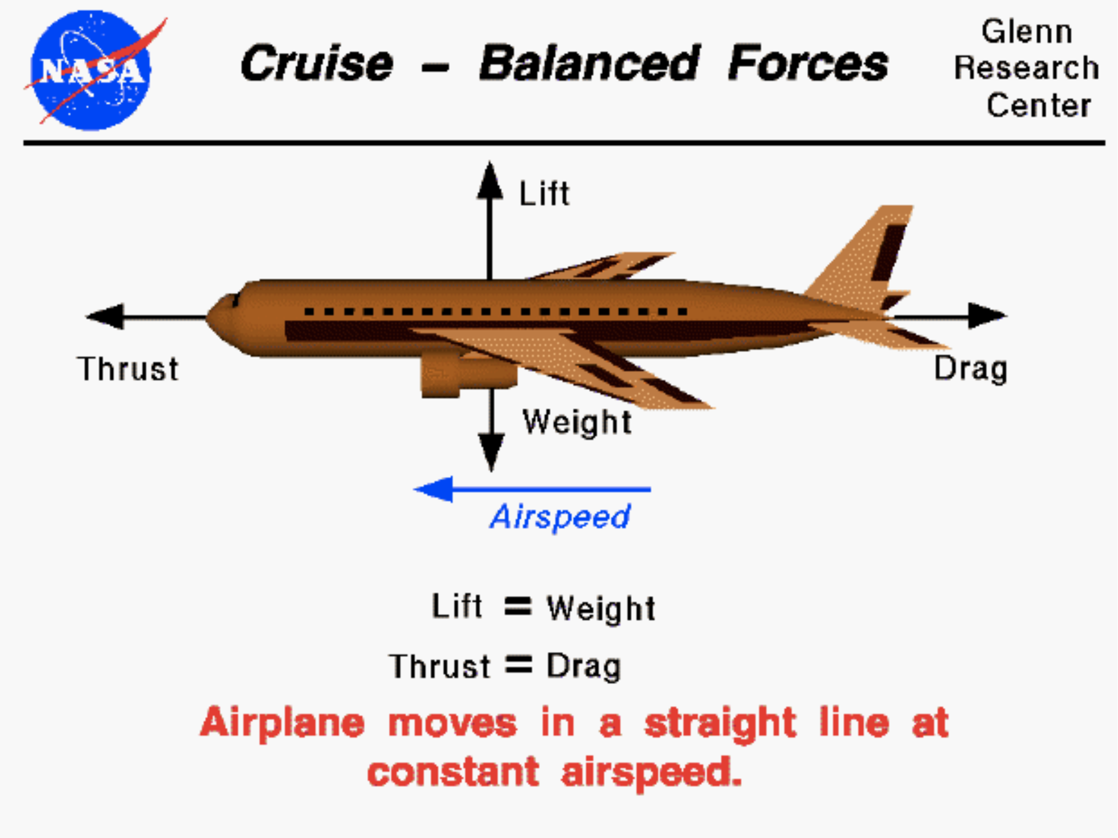


Figure. Force diagram of an aircraft during cruise.

Cruise – Balanced Forces: https://www.grc.nasa.gov/www/k-12/VirtualAero/BottleRocket/airplane/cruise.html

Equations of Motion

$$F_N(t) = D(t)$$

$$W(t) = L(t)$$

Aerodynamic Forces

$$D(t) = q_{cr} A c_D$$

$$L(t) = q_{cr} A c_L$$

$$q_{cr} = (1/2)\rho_{cr} v_{cr}^2$$

$$c_D(t) = c_{D,O} + k c_L(t)^2$$

$$k = \frac{1}{\pi A R \varepsilon}$$

Empirical Relations

$$W(t) = W_0 + W_f(t)$$

$$\frac{dW}{dt} = \frac{dm_f}{dt} g = -c_{j,cr} F_N(t) g$$

$$c_{j,cr} = c_{j,0} + c_{j,1} v_{cr}$$

1st Order Linear ODE

$$\frac{dW}{dt} = k_2 \left(W(t)\right)^2 + k_1$$

Cruising Flight Phase - Results

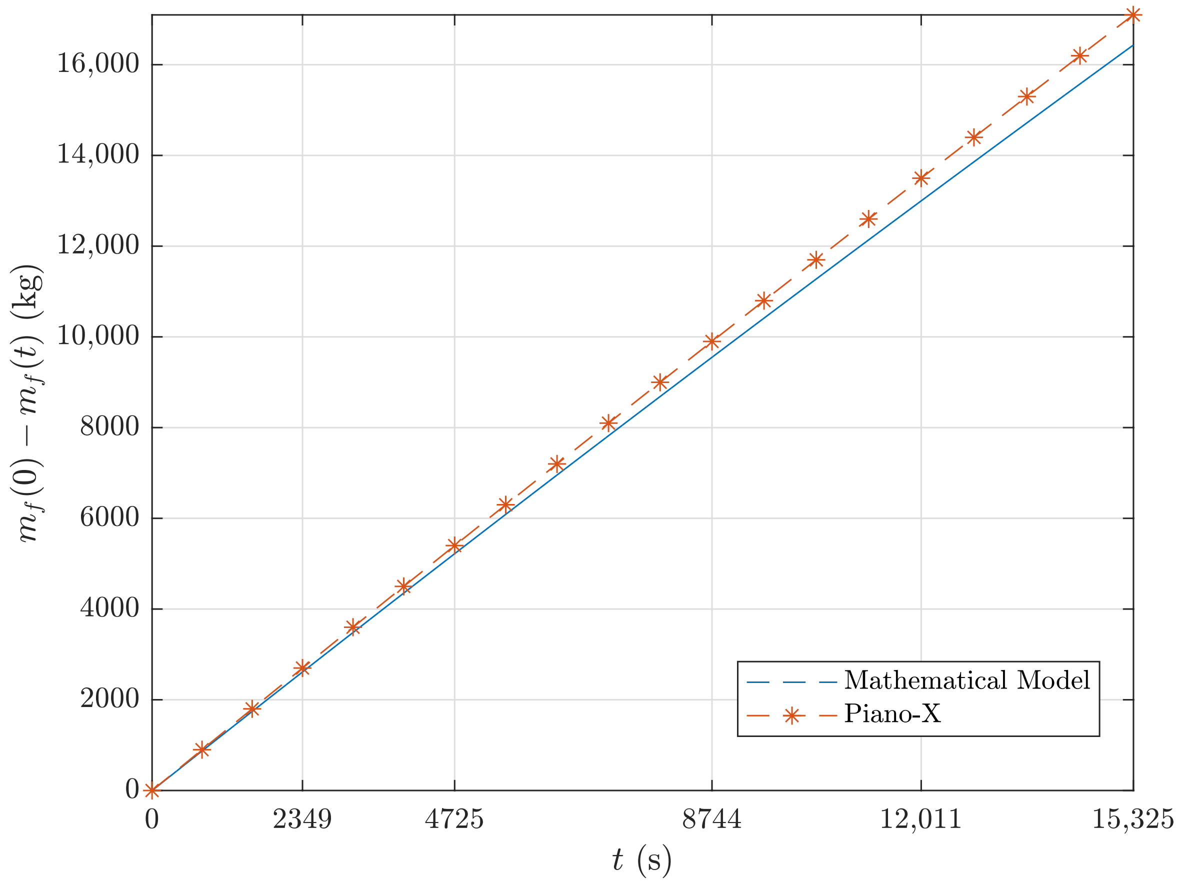


Figure. Fuel consumed during the cruising flight phase.

Closed-Form Solution of the Aircraft's Fuel Consumption

For the initial value problem (IVP) $\mathrm{m_f}(0) = m_{f,0}$:

$$W(t) = W(0)\frac{1-(1/\beta)\tan(c_{j,cr}g\sqrt{c_{D,0}k}t)}{1-(\beta)\tan(c_{j,cr}g\sqrt{c_{D,0}k}t)}$$

$$\beta = \frac{W(0)}{q_{cr}A}\sqrt{\frac{k}{c_{D,0}}}$$

$$m_{f,cr} = m_f(t_{cr}) = 16{,}435 \text{ kg}$$

$$m_{f,Piano-X} = 17{,}115 \text{ kg}$$

Climbing Flight Phase

https://en.wikipedia.org/wiki/Climb_%28aeronautics%29#/media/File:ENTERAIR6-SPENB.jpg

Climbing Flight Phase – Mathematical Model

2nd Order Non-linear ODE

$$\kappa_1 (1 - \alpha h(t))^q \left(\frac{dh}{dt}\right)^{-2} + \kappa_2 (1 - \alpha h(t))^r \left(\frac{dh}{dt}\right)^{-1} + \kappa_3 = \frac{d^2 h}{dt^2}$$

Empirical Relations

$$c_j(h, \eta) = \left(\zeta_1 + \zeta_2 \frac{\eta}{a(h)}\right) (\rho(h))^m$$

$$F_N(h, \eta) = \left(F_1 + F_2 \frac{\eta}{a(h)}\right) (\rho(h))^m$$

Conclusions

Conclusions

- Our closed-form solutions were proven to provide accurate results by comparison with *Piano-X Aircraft Performance and Emissions* (https://www.lissys.uk/PianoX.html) software.
- The closed-form solutions mainly provide:
  - A **closed-form relationship** between the aircraft's fuel consumption and aerodynamic, engine and design parameters,
  - A tool for the accurate quantification of **pollutant gas emissions**.
  - A means of studying the **aircraft's performance** during each flight phase,
  - And enables further **optimization** and **sensibility analyses**.

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F. Velásquez-SanMartín, X. Insausti, M. Zárraga-Rodríguez and J. Gutiérrez-Gutiérrez, "*A Mathematical Model for the Analysis of Jet Engine Fuel Consumption during Aircraft Take-off*," 2022 IEEE Aerospace Conference (AERO), Big Sky, MT, USA, 2022, pp. 1-10, doi:10.1109/AERO53065.2022.9843276

F. Velásquez-SanMartín, X. Insausti, M. Zárraga-Rodríguez, J. Gutiérrez-Gutiérrez, "*A Mathematical Model for the Analysis of Jet Engine Fuel Consumption during Aircraft Cruise*," Energies 2021, 14, 3649. doi:10.3390/en14123649

F. Velásquez-SanMartín, X. Insausti, M. Zárraga-Rodríguez, J. Gutiérrez-Gutiérrez, "*A Mathematical Model for the Analysis of Jet Engine Fuel Consumption during Aircraft Climb*"

Thanks for your attention!

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