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A statistical measure of complexity

R. López-Ruiz [a], H.L. Mancini [b], X. Calbet [c]

[a] *Laboratoire de Physique Statistique, ENS, 24 Rue de Lhomond, Paris, France*
[b] *Departamento de Física, Universidad Privada de Navarra, Pamplona, Spain*
[c] *Instituto de Astrofísica de Canarias, La Laguna, Tenerife, Spain*

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Abstract

A measure of *complexity* based on a probabilistic description of physical systems is proposed. This measure incorporates the main features of the intuitive notion of such a magnitude. It can be applied to many physical situations and to different descriptions of a given system. Moreover, the calculation of its value does not require a considerable computational effort in many cases of physical interest.

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On the most basic grounds, an object, a procedure, or system is said to be "complex" when it does not match patterns regarded as simple. This sounds rather tautological but common knowledge tells us what is simple and complex: simplified systems or idealizations are always a starting point to solve scientific problems. The notion of "complexity" in physics [1,2] starts by considering the perfect crystal and the isolated ideal gas as examples of simple models and therefore as systems with zero "complexity". Let us briefly recall their main characteristics with "order", "information" and "equilibrium".

A perfect crystal is completely ordered and the atoms are arranged following stringent rules of symmetry. The probability distribution for the states accessible to the perfect crystal is centered around a prevailing state of perfect symmetry. A small piece of "information" is enough to describe the perfect crystal: the distances and the symmetries that define the elementary cell. The "information" stored in this system can be considered minimal. On the other hand, the isolated ideal gas is completely disordered. The system can be found in any of its accessible states with the same probability. All of them contribute in equal measure to the "information" stored in the ideal gas. It has therefore a maximum "information". These two simple systems are extrema in the scale of "order" and "information". It follows that the definition of "complexity" must not be made in terms of just "order" or "information".

It might seem reasonable to propose a measure of "complexity" by adopting some kind of distance from the equiprobable distribution of the accessible states of the system. Defined in this way, "disequilibrium" would give an idea of the probabilistic hierarchy of the system. "Disequilibrium" would be different from zero if there are privileged, or more probable, states among those accessible. But this would not work. Going back to the two examples we began with, it is readily seen that a perfect crystal is far from an equidistribution

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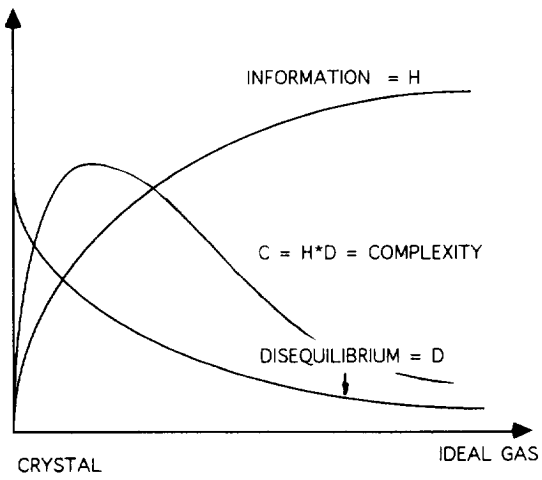

Fig. 1. Sketch of the intuitive notion of the magnitudes of "information" (H) and "disequilibrium" (D) for the physical systems and the behavior intuitively required for the magnitude "complexity". The quantity $C = HD$ is proposed to measure such a magnitude.

among the accessible states because one of them is totally prevailing, and so "disequilibrium" would be maximum. For the ideal gas, "disequilibrium" would be zero by construction. Therefore such a distance or "disequilibrium" (a measure of a probabilistic hierarchy) cannot be directly associated with "complexity".

In Fig. 1 we sketch an intuitive qualitative behaviour for "information" H and "disequilibrium" D for systems ranging from the perfect crystal to the ideal gas. This graph suggests that the product of these two quantities could be used as a measure of "complexity": $C = HD$ (Fig. 1). The function C has indeed the features and asymptotical properties that one would expect intuitively: it vanishes for the perfect crystal and for the isolated ideal gas, and it is different from zero for the rest of the systems of particles. We will follow these guidelines to establish a quantitative measure of "complexity".

Before attempting any further progress, however, we must recall that "complexity" cannot be measured univocally, because it depends on the nature of the description (which always involves a reductionist process) and on the scale of observation. Let us take an example to illustrate this point. A computer chip can look very different at different scales. It is an entangled array of electronic elements at microscopic scale but only an ordered set of pins attached to a black box at a macroscopic scale.

We shall now discuss a measure of "complexity" based on the statistical description of systems. Let us assume that the system has N accessible states $\{x_1, x_2, ..., x_N\}$ when observed at a given scale. We will call this an N-system. Our understanding of the behaviour of this system determines the corresponding probabilities $\{p_1, p_2, ..., p_N\}$ (with the condition $\sum_{i=1}^{N} p_i = 1$) of each state ($p_i \neq 0$ for all i). Then the knowledge of the underlying physical laws at this scale is incorporated into a probability distribution for the accessible states. It is possible to find a quantity measuring the amount of "information". Under the most elementary conditions of consistency, Shannon [3] determined the unique function $H(p_1, p_2, ..., p_N)$ that accounts for the "information" stored in a system,

$$H = -K \sum_{i=1}^{N} p_i \log p_i, \tag{1}$$

where K is a positive constant. The quantity H is called *information*. In the case of a crystal, a state x_c would be the most probable $p_c \sim 1$, and all other x_i would be very improbable, $p_i \sim 0$, $i \neq c$. Then $H_c \sim 0$. On the other side, equiprobability characterizes an isolated ideal gas, $p_i \sim 1/N$ so $H_g \sim K \log N$, i.e., the maximum of information for an N-system. (Notice that if one assumes equiprobability and $K = \kappa \equiv$ Boltzmann constant, H is identified with the thermodynamic entropy ($S = \kappa \log N$)). Any other N-system will have an amount of information between those two extrema.

Let us propose a definition of *disequilibrium* D [4] in an N-system. The intuitive notion suggests that some kind of distance from an equiprobable distribution should be adopted. Two requirements are imposed on the magnitude of D: $D > 0$ in order to have a positive measure of "complexity" and $D = 0$ on the limit of equiprobability. The straightforward solution is to add the quadratic distances of each state to the equiprobability as follows,

$$D = \sum_{i=1}^{N} (p_i - 1/N)^2. \tag{2}$$

According to this definition, a crystal has maximum disequilibrium (for the dominant state, $p_c \sim 1$, and $D_c \to 1$ for $N \to \infty$) while the disequilibrium for an ideal gas vanishes ($D_g \sim 0$) by construction. For any

other system D will have a value between these two extrema.

We now introduce the definition of *complexity* C of an N-system. This is simply the interplay between the information stored in the system and its disequilibrium,

$$C = HD = -\left(K\sum_{i=1}^{N} p_i \log p_i\right)\left(\sum_{i=1}^{N}(p_i - 1/N)^2\right). \quad (3)$$

This definition fits the intuitive arguments. For a crystal, disequilibrium is large but the information stored is vanishingly small, so $C \sim 0$. On the other hand, H is large for an ideal gas, but D is small, so $C \sim 0$ as well. Any other system will have an intermediate behavior and therefore $C > 0$.

As was intuitively suggested, the definition of complexity (3) also depends on the *scale*. At each scale of observation a new set of accessible states appears with its corresponding probability distribution so that complexity changes. Physical laws at each level of observation allow us to infer the probability distribution of the new set of accessible states, and therefore different values for H, D and C will be obtained. (In the most complicated situations, where there exist extremely many different states, there are methods to calculate functions of the probability distribution [6].) The passage to the case of a continuum number of states, x, is straightforward. Thus we must treat with probability distributions with a continuum support, $p(x)$, and normalization condition $\int_{-\infty}^{\infty} p(x)\,dx = 1$. Disequilibrium has the limit $D = \int_{-\infty}^{\infty} p^2(x)\,dx$ and the complexity is defined by

$$C = HD = -\left(K\int_{-\infty}^{\infty} p(x)\log p(x)\,dx\right) \times \left(\int_{-\infty}^{\infty} p^2(x)\,dx\right). \quad (4)$$

Direct simulations of the definition give the values of C for general N-systems. The set of all the possible distributions $\{p_1, p_2, \ldots, p_N\}$ where an N-system could be found is sampled. For the sake of simplicity H is normalized to the interval $[0,1]$. This magnitude is called $\overline{H}$. Thus $\overline{H} = \sum_{i=1}^{N} p_i \log p_i / \log N$. For each distribution $\{p_i\}$ the normalized information $\overline{H}(\{p_i\})$, and the disequilibrium $D(\{p_i\})$ (Eq. (2)) are calculated. In each case the normalized complexity $\overline{C} = \overline{H}D$ is obtained and the pair $(\overline{H}, \overline{C})$ stored. These two magnitudes are plotted on a diagram $(\overline{H}, \overline{C}(\overline{H}))$ in order to verify the qualitative behavior predicted in Fig. 1. For $N = 2$ an analytical expression for the curve $\overline{C}(\overline{H})$ is obtained. If the probability of one state is $p_1 = x$, that of the second one is simply $p_2 = 1 - x$. The complexity of the system will be

$$\overline{C}(x) = \overline{H}(x)D(x) = -\frac{1}{\log 2}\left[x\log\left(\frac{x}{1-x}\right) + \log(1-x)\right] \times 2(x - \tfrac{1}{2})^2. \quad (5)$$

Complexity as a function of $\overline{H}$ is shown in Fig. 2a. It vanishes for the two simplest two-systems: the crystal ($\overline{H} = 0$; $p_1 = 1$, $p_2 = 0$) and the ideal gas ($\overline{H} = 1$; $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$). Let us notice that this curve is the simplest one that fulfills all the conditions discussed in the introduction. The largest complexity is reached for $\overline{H} \sim \frac{1}{2}$ and its value is: $C(x \sim 0.11) \sim \overline{C}\log 2 \sim 0.105$. For $N > 2$ the relationship between $\overline{H}$ and $\overline{C}$ is not univocal anymore. Many different distributions $\{p_i\}$ store the same information $\overline{H}$ but have different complexity $\overline{C}$. Fig. 2b displays such a behavior for $N = 3$. If we take the maximum complexity $\overline{C}_{\max}(\overline{H})$ associated with each $\overline{H}$ a curve similar to the one shown in a two-system is recovered. Every three-system will have a complexity below this line. In Fig. 2a curves $\overline{C}_{\max}(\overline{H})$ for the cases $N = 3, 5, 7$ are also shown. Let us observe the shift of the complexity-curve peak to smaller values of entropy for rising N. This fact agrees with the intuition telling us that the biggest complexity (number of possibilities of "complexification") be reached for lesser entropies for the systems with bigger number of states.

Also it is possible to compute these quantities in other relevant physical situations and in continuum systems.

We can now go one step further. The most important point is that the new definition should work in systems out of equilibrium. We use two examples of such systems where it is known that very complex dy-

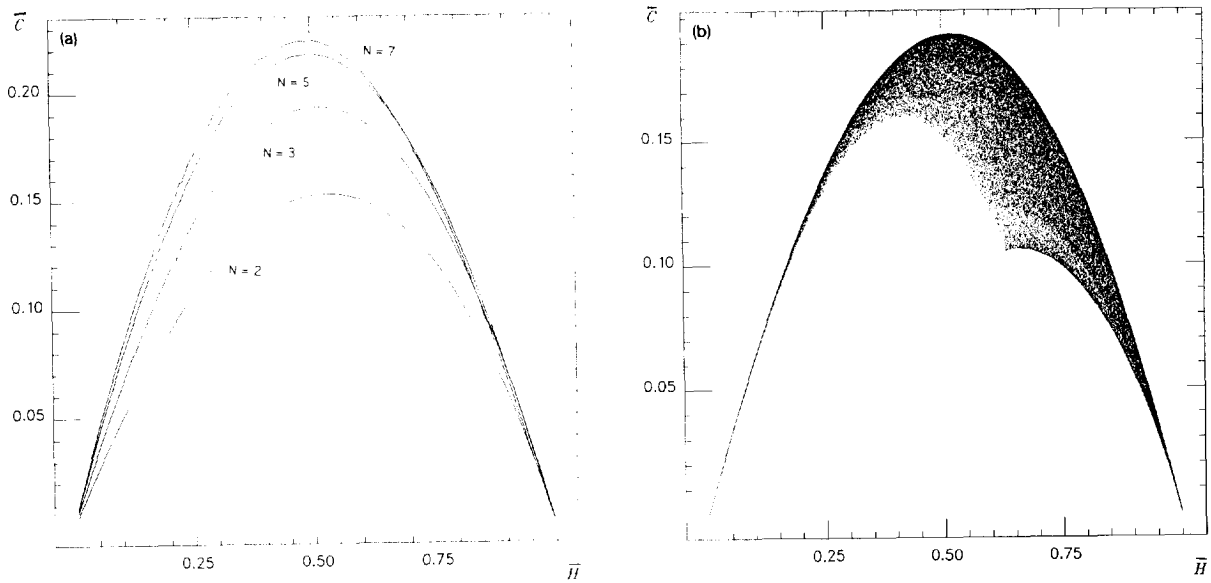

Fig. 2. (a) Complexity ($\overline{C} = \overline{H}D$) as a function of the information ($\overline{H}$) for a system with two accessible states ($N = 2$). Also curves of maximum complexity ($\overline{C}_{max}$) are shown for the cases: $N = 3, 5, 7$. (b) In general, the dependence of complexity ($\overline{C}$) on information ($\overline{H}$) is not univocal: many distributions $\{p_i\}$ can present the same value of $\overline{H}$ but different $\overline{C}$. This is shown in the case $N = 3$.

namics could show up. They are the logistic map and the "Lorenz" map.

Logistic map. The mapping $x_{n+1} = \alpha x_n(1 - x_n)$, $\alpha \in [0, 4]$, is a well known chaotic system. There are two points in this system where the behaviour is extremely "complex". The first is the accumulation point of the subharmonic cascade. The second is the transition point from chaos to a period three orbit via intermittency [5]. We are going to discuss and study the complexity in the second case only, which corresponds to parameter values around $\alpha_t \sim 3.8284$. Complexity must increase as $\alpha - \alpha_t \to 0$ when $\alpha < \alpha_t$ because the closer we get to the critical point α_t the more improbable and unpredictable is the firing and development of the intermittent bursts. On the contrary, when $\alpha > \alpha_t$ the system becomes periodic and does not have any complexity. Complexity[1] undergoes a rapid increase in the intermittency region and a sharp transition to zero at the transition point (Fig. 3a). (This resembles the curve of the specific heat as a function of temperature in a second order phase transition.)

Lorenz map. Let us consider a mapping that mimics the behavior and development of the Lorenz attractor. This is the simplest mapping that includes the main features of the first return map of this attractor: $\{x_{n+1} = \alpha x_n$ if $x_n < 0.5$ and $x_{n+1} = \alpha(x_n - 1) + 1$ if $x_n > 0.5\}$ where $\alpha \in (0, 2)$. Its dynamic evolution displays three different behaviors: (1) R_1: $\alpha \in (0, 1)$. The system goes to a fixed point (0 or 1) and the output is a constant signal; (2) R_2: $\alpha \in (1, 2)$. A variable chaotic attractor is present in this whole region and a chaotic signal is obtained; (3) R_3 : $\alpha \approx 2$. The Bernoulli shift is reached and the output signal is random. Results for complexity are given in Fig. 3b. In region R_1, C vanishes because there is nothing to explain a constant signal. In region R_2, $C \neq 0$ and shows a complicated dependence on α. Variations in C are due to changes in the structure of the underlying chaotic attractor. When $\alpha \to 2$ (region R_3) a random system is reached, and again $C = 0$.

[1] For each parameter value, α, calculations of complexity have been done in the following steps: (i) the dynamics is reduced to a binary sequence (its symbolic dynamics: 0 if $x < 0.5$ and 1 if $x > 0.5$); (ii) the scale n in these cases is the length n of the binary strings considered as states of the system. Thus, 2^n states are possible in this case; (iii) the probability of each state is measured when the system is running over 2^{n+10} iterations, that is binary sequences of 2^{n+10} digit-length; (iv) H, D and C are calculated directly with that probability distribution.

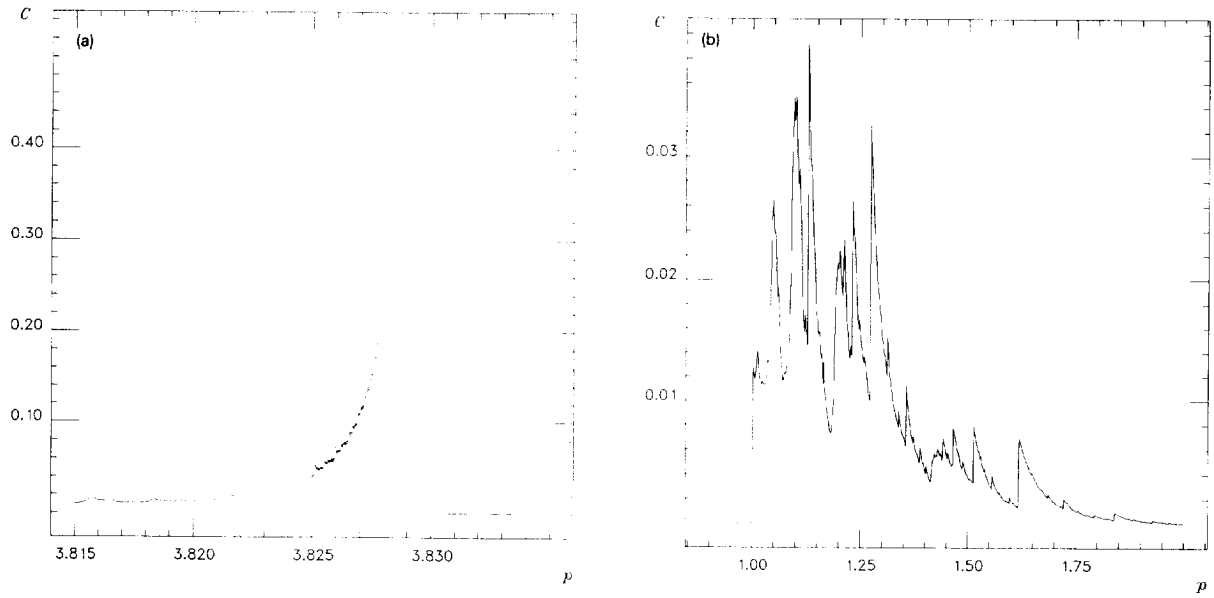

Fig. 3. (a) Behavior of complexity C in the transition point ($\alpha_t \sim 3.8284$) where the system (logistic map) goes from a chaotic dynamics ($\alpha < \alpha_t$) to a period three orbit ($\alpha > \alpha_t$) by intermittency. (b) Complexity displayed by the "Lorenz" map for a region of the parameter space $\alpha \in [0, 2]$. (See explanation in the text.) (Calculations have been done for a scale $n = 12$, $n \equiv$ length of the binary strings analyzed.)

Let us return to the point at which we started the initial discussion. Any notion of complexity in physics [1,2] should only be made on the basis of a well defined or operational magnitude. But two additional requirements are needed in order to obtain a good definition of complexity in physics: (c1) the new magnitude must be measurable in many different physical systems and (c2) a comparative relationship and a physical interpretation between any two measurements should be possible.

Many different definitions of complexity have been proposed to date, mainly in the realm of computational sciences. Among these, several can be cited: algorithmic complexity (Kolmogorov–Chaitin) [7,8], the Lempel–Ziv complexity [9], the logical depth of Bennett [10], the effective measure complexity of Grassberger [11], the complexity of a system based in its diversity [12], the thermodynamical depth [13], complexities of formal grammars, etc. The definition of complexity proposed in this work offers a new point of view, based on a statistical description of systems at a given *scale*. In this scheme the knowledge of the physical laws governing the dynamic evolution in that scale is used to find its accessible states and its probability distribution. This process would immediately indicate the value of complexity. In essence this is nothing but an interplay between the information stored by the system and the *distance from equipartition* (measure of a probabilistic hierarchy between the observed parts) of the probability distribution of its accessible states . Besides giving the main features of an "intuitive" notion of complexity, we showed that it successfully enables us to discern situations regarded as complex, both for a local transition (Fig. 3a) and for a global behavior (Fig. 3b) in systems out of equilibrium.

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References

[1] G. Parisi, Phys. World, Vol. 6, No. 9 (1993).
[2] P.W. Anderson, Physics Today, July (1991).
[3] C.E. Shannon and W. Weaver, The mathematical theory of communication (University of Illinois Press, Urbana, IL, 1949).

[4] G. Nicolis and I. Prigogine, Self-organisation in non-equilibrium systems (Wiley, New York, 1977).
[5] Y. Pomeau and P. Manneville, Commun. Math. Phys. 74 (1980) 189.
[6] T. Poschel, W. Ebeling and H. Rosé, J. Stat. Phys. 80 (1995) 1443.
[7] A.N. Kolmogorov, Probl. Inform. Theory 1 (1965) 3.
[8] G.J. Chaitin, Information, randomness and incompleteness (World Scientific, Singapore, 1990).
[9] A. Lempel and J. Ziv, IEEE Trans. Inform. Theory 22 (1976) 75.
[10] C.H. Bennett, in: Emerging syntheses in science, ed. D. Pines (Addisson-Wesley, Reading, MA, 1988).
[11] P. Grassberger, Int. J. Theor. Phys. 25 (1986) 907.
[12] B.A. Huberman and T. Hogg, Physica D 22 (1986) 376.
[13] S. LLoyd and H. Pagels, Ann. Phys. (NY) 188 (1988) 186.