International Journal of Bifurcation and Chaos, Vol. 10, No. 4 (2000) 829–833


# PHASE CLUSTERING AND COLLECTIVE BEHAVIORS IN GLOBALLY COUPLED MAP LATTICES DUE TO MEAN FIELD EFFECTS

D. MAZA, S. BOCCALETTI and H. MANCINI
*Department of Physics and Applied Mathematics,*
*Universidad de Navarra, Irunlarrea s/n, 31080 Pamplona, Spain*

Received May 14, 1999; Revised September 6, 1999

We describe the emergence of phase clustering and collective behaviors in an ensemble of chaotic coupled map lattices, due to a mean field interaction. This kind of interaction is responsible for the appearence of a collective state, wherein the mean field evolution of each lattice undergoes a periodic behavior in space. We analyze the transition to such a state in an ensemble of one-dimensional lattices of logistic maps, showing that the resulting behavior cooperatively maximizes the energy of the mean field activity.

## 1. Introduction

In the last decade, synchronization of chaos has attracted a lot of interest within the scientific community. For concentrated systems, chaos synchronization refers to a process wherein the trajectories of two systems are linked to the same values at the same times, so that the same systems remain in step with each other. This mechanism was originally introduced by Pecora and Carroll (PC) [Pecora & Carroll, 1990], for two identical chaotic dynamics evolving from different initial conditions. By means of a unidirectional coupling, PC have shown that the two dynamics can be synchronized, provided that the sub-Lyapunov exponents of the subsystem to be synchronized are all negative.

One of the main uses of the PC idea was to produce a secure communication between a message sender and a message receiver [Cuomo & Oppenheim, 1993; Gershenfeld & Grinstein, 1995; Kocarev & Parlitz, 1995; Peng *et al.*, 1996; Boccaletti *et al.*, 1997] when applied in connection with the possibility of encoding a message within a chaotic dynamics [Hayes *et al.*, 1994].

More recently, the concept of chaos synchronization has been extended to that of phase synchronization of chaotic systems [Rosenblum *et al.*, 1996]. In this case, the interaction of nonidentical chaotic systems can produce a perfect locking of their phases, while their amplitudes remain uncorrelated. This phenomenon and the transition to it has been largely studied for two coupled oscillators with reference to the Rössler system [Rosenblum *et al.*, 1997; Rosa *et al.*, 1998].

Another type of synchronization was later called lag synchronization, consisting of the fact that the two outputs coming from two nonidentical oscillators become identical in phases and amplitudes, but shifted in a lag time of $\tau_{\text{lag}}$ [Rosenblum *et al.*, 1997].

Finally, a further type of synchronization feature was called generalized synchronization, implying the hooking of the output of one system to a given function of the output of the other [Rulkov *et al.*, 1995; Kocarev & Parlitz, 1996].

While most of the above results refer to synchronization behavior of confined systems, i.e. systems modeled by ordinary differential equations, the synchronization effects in large populations of coupled chaotic dynamical units are currently a subject of active investigations [Strogatz *et al.*, 1992; Heagy *et al.*, 1995; Zanette, 1997].

In this paper we present a case of this latter situation, and show how a global coupling in a ensemble of nonidentical chaotic map lattices can induce the appearance of a collective state, wherein the behavior of the mean fields shows a phase clustering. We then characterize the transition to such a state, and extract the main features of the resulting dynamics.

Let us begin by considering an ensemble of $N$ coupled map lattices, each one formed by $L$ logistic maps. In this system, the state $x_k^i$ of the $k$th map ($k = 1, \ldots, L$) of the $i$th lattice ($i = 1, \ldots, N$) evolves at time $n+1$ through the rule

$$x_k^i(n+1) = (1-2\varepsilon)\mathcal{F}_k^i(x_k^i(n)) + \varepsilon\mathcal{F}_k^i(\mathcal{M}^{i-1}(n)) + \varepsilon\mathcal{F}_k^i(\mathcal{M}^{i+1}(n))\,. \qquad (1)$$

In Eq. (1), $\varepsilon$ is a real coupling parameter, $\mathcal{F}_k^i(x) = \mu_k^i x(1-x)$ is the logistic map, and $3.569946 < \mu_k^i \le 4$ are $N \times L$ randomly distributed parameters (with flat distribution). The condition $3.569946 < \mu_k^i \le 4$ implies that most of the maps are considered within the chaotic regime. However, a portion of them can occasionally lie within some periodic window. Furthermore $\mathcal{M}^i(n) \equiv 1/L\sum_{k=1}^{L} x_k^i(n)$ is the mean activity of the $i$th lattice at time $n$, and we consider explicitly periodic boundary conditions along direction $i$ and randomly selected initial conditions $x_k^i(0)$ for each map.

When $\varepsilon = 0$, Eq. (1) describes the dynamics of $N \times L$ independent nonidentical logistic maps.

When, instead, $\varepsilon \neq 0$, this implies a global coupling among the lattices which equally distributes on each element of one lattice the mean activity of the nearest lattices.

For the time being, we select $\mu_k^i$ as randomly distributed between 3.8 and 4.0 (with flat distribution).

In order to quantitatively characterize the appearance of phase clustering in system (1), we need to define a *distance in phase* $D_{i,j}$ between the mean fields of the $i$th and $j$th lattices. We then consider the signals $\mathcal{M}^i(t)$ and $\mathcal{M}^j(t)$, and take the former one as a reference signal for the phase. Let us suppose that $\mathcal{M}^i(\tau_n)$ displays a $n$th local minimum (maximum) at a given time $\tau_n$. At the same time, we also check whether $\mathcal{M}^j(\tau_n)$ is a local minimum (maximum). If the above condition does not hold, this means that, at $t = \tau_n$, the two signals do not belong to the same phase cluster, and we add one to their phase distance ($D_{i,j}(n+1) = D_{i,j}(n)+1$). In the opposite case, the two signals are clustered in phase, and their phase distance is left unchanged. Then we look for the next local minimum (maximum) at time $\tau_{n+1}$. After having traveled through the signal $\mathcal{M}^i(t)$, and after having repeated the same procedure taking $\mathcal{M}^j(t)$ as phase reference signal, the final value of $D_{i,j}$ comes out to be an integer number ranging from zero (perfect phase clustering) to $N_i+N_j$ (signals completely unclustered), $N_i(N_j)$ being the total number of local extrema of the $i$th ($j$th) lattice.

As a first step of our investigation, we are interested in looking for the emergence of phase clustering phenomena, as $\varepsilon$ increases. Looking at system (1), one easily realizes that, besides ruling the strength of the coupling, the $\varepsilon$ parameter also renormalizes each $\mu_k^i$ value. Therefore, in order to fully appreciate the effects of the mean field coupling in the process of phase clustering, we should compare the evolution of system (1), with the evolution of the following uncoupled system

$$x_k^i(n+1) = \mathcal{F}_{(\varepsilon),k}^i(x_k^i(n))\,, \qquad (2)$$

with the same parameters, boundary conditions and initial conditions as for system (1), and with $\mathcal{F}_{(\varepsilon),k}^i(x) = \mu_k^i(\varepsilon)x(1-x)$, $\mu_k^i(\varepsilon) = \mu_k^i(1-2\varepsilon)$. Now, it is a known result that the evolution of the mean field of each ensemble $i$ of uncoupled maps in system (2), is periodic (period 2) in the limit $L \to \infty$ as far as a finite portion of the maps of the ensemble is beyond the band-merging point between chaotic bands 1 and 2 of the logistic map. Such a behavior, in our case, would occur for $3.8(1-2*\varepsilon) \simeq 3.678$, that is for $\tilde{\varepsilon} \simeq 0.016$. This fact, in our formalism, would give rise to a trivial phase clustering. Indeed, since the mean fields would constitute a bunch of period 2 signals, they would result mutually either in phase or out of phase.

However, the above said is valid only when approaching the limit $L \to \infty$, whereas, for finite $L$ values, system (2) shows the dynamics reported in Fig. 1(a). In Fig. 1(a) the normalized phase distances $D$ (0 means in phase, 1 means in antiphase) are shown versus $\varepsilon$ for $L = N = 100$. The formation of the two phase clusters is the result of a very slow diffusion process of the phase distances when increasing $\varepsilon$, and occurs very far away from the expected transition point $\tilde{\varepsilon}$. Figure 1(b) reports the same results, but coming from the simulation of system (1) in the same conditions as Fig. 1(a). Here, as opposed to the previous case, one can easily appreciate the role of the mean field coupling in

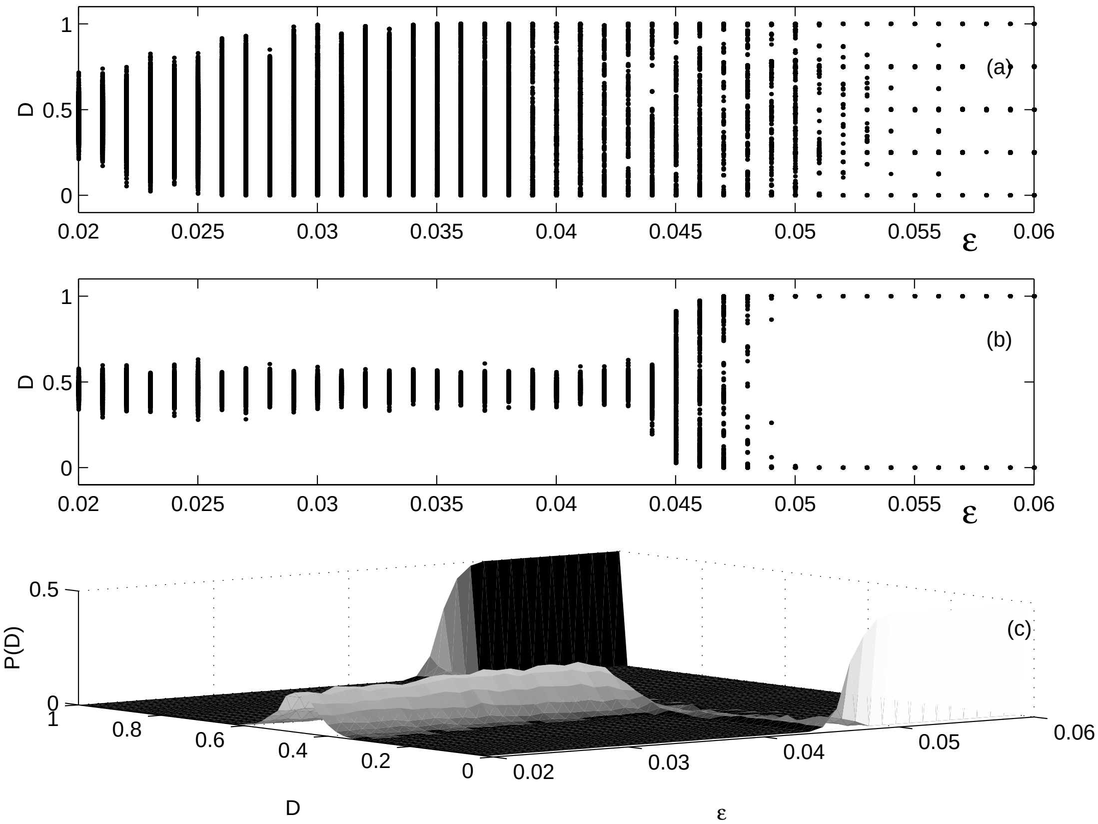


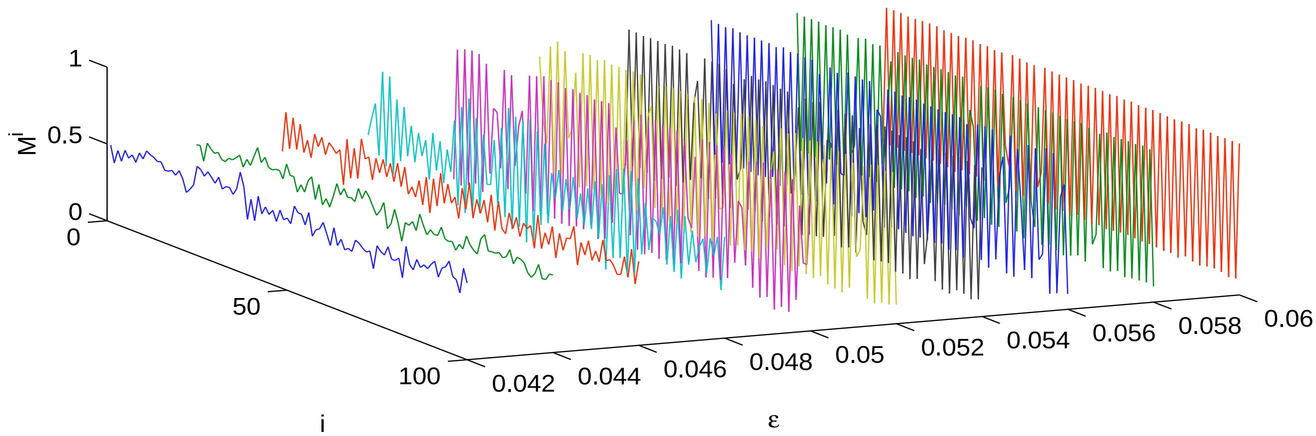

Fig. 1. (a) Normalized phase distances $D$ versus $\varepsilon$ for system (2) obtained from random initial conditions. A slow diffusion process in the phase distances leads eventually to the formation of a phase clustered state. (b) Same for system (1). Notice the sharp transition to a phase clustered state occurring now at $\epsilon \simeq 0.046$. (c) Three-dimensional graph reporting the probability distribution $P(D)$ of the phase distances as a function of $\varepsilon$ for system (1).

producing a sharp transition towards a phase clustered state, occurring at $\varepsilon \simeq 0.046$. The sharpness of the transition toward a phase clustered state is highlighted in Fig. 1(c), where we report the value of the probability distribution $P(D)$ of the values of the observed phase distances, as a function of $\varepsilon$. The appearance of such a phase clustered state was reported by us in [De San Roman *et al.*, 1998].

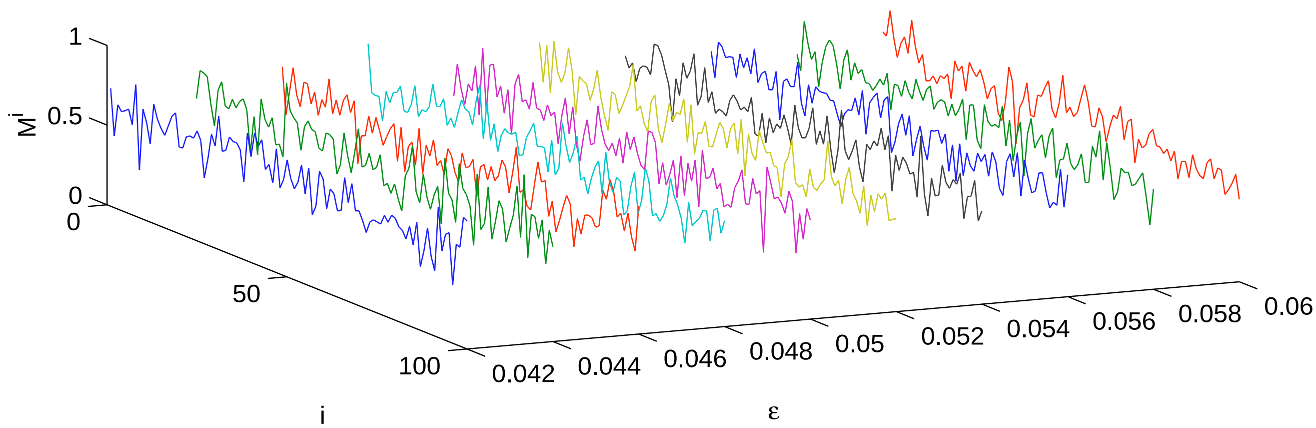
Up to now, we have only demonstrated that the mean field coupling enhances the transition toward a collective phase clustered state, also for finite ensembles. In the following, we will proceed towards the next step of our analysis, which will demonstrate that this state is the result of a cooperative process, and, in this sense, is qualitatively different from that obtained in the case of uncoupled maps. For this purpose, let us consider more deeply the dynamics of system (1) close to the transition point $\varepsilon \simeq 0.046$, and focus on the spatial features along direction $i$.

We then let system (1) evolve at a given $\varepsilon$ from random initial conditions for a while sufficient to end up the transient before the system reaches the asymptotic behavior, and consider all mean fields $\mathcal{M}^i$, $i = 1, \ldots, N$, as functions of time. At each time $t_n$, it is possible to code these mean fields into a phase pattern $\mathcal{P}^i(t_n)$, by the following procedure. Suppose that at time $t_n$ $\mathcal{M}^1(t_n)$ shows a local maximum (minimum). We then conventionally fix $\mathcal{P}^1(t_n) = 0$, and recur all the other mean fields $i \neq 1$. If $\mathcal{M}^i(t_n)$ also shows a local maximum (minimum), we fix $\mathcal{P}^i(t_n) = 0$, otherwise we put $\mathcal{P}^i(t_n) = 1$. This way, one constructs a binary phase pattern $\mathcal{P}^i(t_n)$, which holds information on how (in the space along direction $i$) the mean fields arrange in phase to realize the phase clustered state.

The results show that, while for system (2) the emergence of the trivial phase clustering is associated with a nonperiodic phase patterning $\mathcal{P}^i$, the transition point $\varepsilon \simeq 0.046$ for system (1) corresponds to the emergence of a periodic (period 2) patterning $\mathcal{P}^i$. This implies that, in the phase clustered state, the different map lattices




(a)




(b)

Fig. 2. Three-dimensional plots of the mean fields $\mathcal{M}^i$ (at a fixed time) as functions of $i$ and $\varepsilon$ for (a) system (1) and (b) system (2). In (a) a spatial period 2 appears when crossing $\varepsilon \simeq 0.046$, and in (b) one cannot appreciate qualitative changes in the behavior of the $\mathcal{M}$'s values for the studied range of coupling strength. In both cases the color code has been chosen only to make more understandable the appearance of the spatial structure.

collectively organize in space so as each given lattice $i$ is surrounded by two nearest lattices ($i-1$ and $i+1$) which are in antiphase with respect to it. It is important to remark that the two lattices $i-1$ and $i+1$ are the only ones entering the equations for the lattice $i$. Therefore, we are effectively in the presence of a synchronization phenomenon, which, besides clustering in phase the different lattices, realizes a spatial configuration in which the coupling to the lattices lying within a given phase cluster comes out from the two adjacent lattices which lie within the other phase cluster.

The above said is illustrated in Fig. 2, where we report the three-dimensional plot of the snapshots (at a fixed time) of the mean fields $\mathcal{M}^i$ as functions of $i$ and $\varepsilon$ for both systems (1) (upper plot) and (2) (lower plot), close to the transition point $\varepsilon \simeq 0.046$. Looking at Fig. 2, one easily realizes that there are no qualitative changes in the behavior of system (2) when increasing $\varepsilon$, whereas it is evident that there is a formation of a spatial period 2 in the behavior of the mean fields when crossing $\varepsilon \simeq 0.046$.

The main consequence of the above phenomenon is that the coupling induces a cooperative effect on the mean field activities. In order to quantify this feature, let us define the energy of the mean field produced by each lattice $i$ as follows

$$\mathcal{E}(i) = \frac{1}{S}\sum_{p=1}^{S} |\mathcal{M}^i(t_p)|^2 \,, \tag{3}$$

that is, averaging over $S$ consecutive time evolutions of the system the modulus square of the mean field.

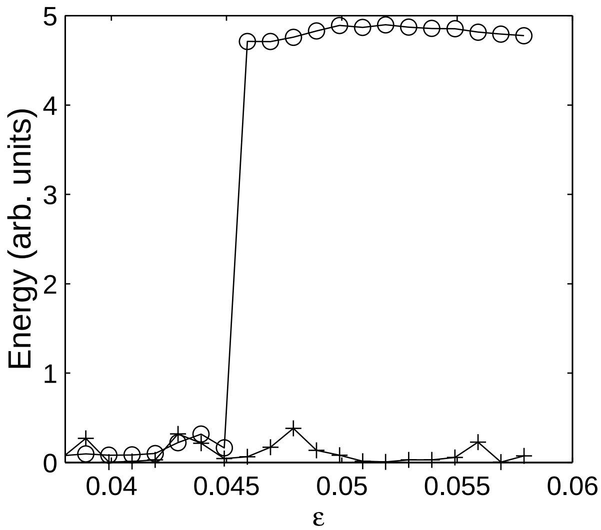



Fig. 3. Mean energy (see text for definition) of the lattice $i = 50$ in both coupled (open circles) and uncoupled (crosses) cases. The cooperative effects of the coupling can be fully appreciated in the sharp rise of the mean field energy for system (1).

Figure 3 reports the behavior of $\mathcal{E}(50)$ as a function of $\varepsilon$ for both systems (1) and (2). Here, the cooperative nature of the mean field coupling appears evident when considering that there is a sharp transition in the energy for system (1) when crossing $\varepsilon \simeq 0.046$, whereas no qualitative changes are appreciated for the uncoupled case.

In conclusion, we have shown that the evolution of a collection of nonidentical globally coupled map lattices leads to a transition toward a phase clustered state. At variance with the trivial clustering which would occur in the uncoupled case, this collective phase clustered state is realized through the emergence of a well-defined periodic phase pattern, inducing a cooperative effect in the coupling factor, which maximizes the mean energy of all the lattices.

## Acknowledgments

The authors acknowledge F. S. de San Roman, A. S. Mikhailov, D. H. Zanette, H. Chaté and A. Lemaitre for fruitful discussions. The work was partly supported by Ministerio de Educacion y Ciencia, Spain (Grant N. PB95-0578), Universidad de Navarra, Spain (PIUNA), Integrated Action Italy-Spain HI97-30. S. Boccaletti acknowledges financial support from EU Contract n. ERBFMBICT983466.

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