Frequency entrainment of nonautonomous chaotic oscillators

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We give evidence of frequency entrainment of dominant peaks in the chaotic spectra of two coupled chaotic nonautonomous oscillators. At variance with the autonomous case, the phenomenon is here characterized by the vanishing of a previously positive Lyapunov exponent in the spectrum, which takes place for a broad range of the coupling strength parameter. Such a state is studied also for the case of chaotic oscillators with ill-defined phases due to the absence of a unique center of rotation. Different phase synchronization indicators are used to circumvent this difficulty.

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Synchronization of coupled chaotic systems has attracted an increasing interest in recent years [1], insofar as several natural observations [2,3] and controlled laboratory experiments [4] have pointed out its ubiquitousness and relevance in nonlinear science.

In particular, phase synchronization (PS) of chaotic oscillators refers to a process whereby a weak coupling makes the phases of the interacting systems evolve in step with each other, even when the corresponding amplitudes are only feebly correlated [5]. More precisely, $m{:}n$ PS corresponds to a situation where the lifts of the two phases to the real line ψ_1 and ψ_2 satisfy $|\psi_2-(m/n)\psi_1|<C$, with C being a positive constant, thus indicating that the coupled oscillators evolve with a $m{:}n$ bounded phase difference.

PS is closely related to the presence of two distinct self-sustained oscillators whose original different rhythms are adjusted by the coupling. This fact made that such studies were so far mostly limited to the autonomous case, where PS was shown to occur in correspondence to the setting of a contractive direction for the phase difference, which occurs when a zero Lyapunov exponent in the spectrum takes a negative value as the coupling strength is increased [5].

In this paper we show that a completely different scenario emerges for chaotic nonautonomous oscillators. Here, all zero Lyapunov exponents are insensitive to the coupling, and a frequency entrainment of dominant peaks in the chaotic spectra occurs in correspondence to a previously positive Lyapunov exponent that vanishes over a broad range of the coupling strength parameter. This indicates that the rhythm adjustment process here takes place also in the absence of a contractive direction for the phases.

To demonstrate the phenomenon, we will refer to a pair of forced Van der Pol oscillators [6] $[\ddot{x}_{1,2}-A\dot{x}_{1,2}(1-x_{1,2}^2)+Bx_{1,2}^3=C\sin(\omega_{1,2}t)]$ in a bidirectional symmetrical coupling configuration. The equations of motion read

$$\dot{x}_{1,2}=y_{1,2},$$

$$\dot{y}_{1,2}=A_{1,2}y_{1,2}(1-x_{1,2}^2)-Bx_{1,2}^3+C\sin(\omega_{1,2}z)+\epsilon(x_{2,1}-x_{1,2}),$$

$$\dot{z}=1, \tag{1}$$

where the subscripts 1 and 2 refer to oscillator 1 and 2, respectively, dots denote temporal derivatives, $A_1=0.6, A_2=0.2, B=1, C=2, \omega_1=0.6$, and $\omega_2=0.65$, are parameters chosen in order to produce a chaotic dynamics for both uncoupled oscillators. In the following we will mainly concentrate on the influence of the coupling parameter ϵ on the dynamics, while the influence of the other parameters will be presented elsewhere. All numerical integrations are performed by means of a fourth-order Runge-Kutta algorithm with integration time step $\delta t=5\times10^{-3}$.

In the uncoupled case and for the selected values of parameters, the two oscillators exhibit a chaotic motion developing onto an attractor which does not display a unique center of rotation [see Fig. 1(a)]. At weak coupling, the two distinct forcing frequencies ω_1 and ω_2 prevent frequency and phase synchronization, insofar as both oscillators will show a strong component of these frequencies in their Fourier spectra. Intermediate couplings [$\epsilon=1$ in Fig. 1(b)] produce a slightly distorted attractor in phase space, which however does not significantly changes the qualitative features of

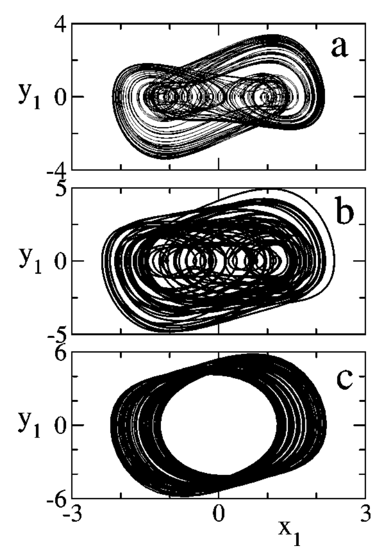

FIG. 1. (y_1,x_1) projection of the attractor of Eqs. (1) for (a) $\epsilon=0$ (uncoupled case), (b) $\epsilon=1$ (intermediate coupling), and (c) $\epsilon=2.7$ (strong coupling). Other parameters specified in the text.

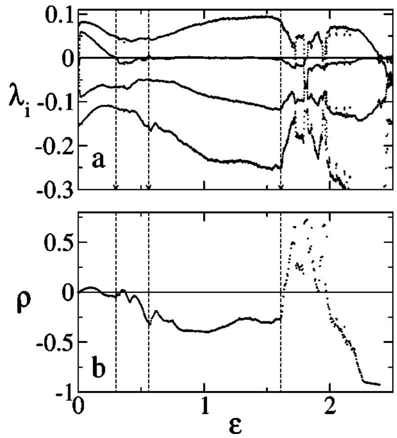

FIG. 2. (a) Lyapunov exponents of Eqs. (1) vs ϵ, (b) linear cross correlation (Pearson's coefficient) between x_1 and x_2 vs ϵ.

the original chaotic motion. Finally, a stronger coupling [ϵ=2.7 in Fig. 1(c)] has the effect of destroying the chaotic attractor and transforming it into a quasiperiodic one. The coupling induces a suppression of chaos [7] and is associated with the signals $x_1(t)$ and $x_2(t)$ being in antiphase. This situation stems from the fact that both system have different forcing frequencies and therefore synchronization in phase is not possible.

As a first task, we make use of the standard analysis tools for the evaluation of Lyapunov exponents in Eqs. (1) [8]. The results are reported in Fig. 2(a), where the vertical dashed lines are used as a guide for a better visualization of the different regimes. At ϵ=0, the spectrum is composed of two positive, two negative, and one zero exponents. This latter one corresponds to the equation $\dot{z}=1$ (and therefore is insensitive to the coupling), accounting for the invariance of Eqs. (1) with respect to time translations.

As ϵ increases, one of the originally positive exponents decreases to a slightly negative value in the range $0.3<\epsilon<0.56$. For $0.56<\epsilon<1.61$, this exponent becomes zero, and eventually takes a negative value for $\epsilon>1.61$. Finally, for $\epsilon>2.4$ no positive Lyapunov exponents are present in the spectrum, indicating that chaos has been suppressed and the signals $x_1(t)$ and $x_2(t)$ are sitting on a quasiperiodic attractor.

Another insight into the synchronization process can be gathered by comparing Fig. 2(a) with the evolution of the linear cross-correlation coefficient (or Pearson's coefficient) between the two temporal series $x_1(t)$ and $x_2(t)$, given by

$$\rho=\frac{\langle(x_1-\langle x_1\rangle)(x_2-\langle x_2\rangle)\rangle}{(\sqrt{\langle(x_1-\langle x_1\rangle)^2\rangle}\sqrt{\langle(x_2-\langle x_2\rangle)^2\rangle}},$$

and reported in Fig. 2(b). One clearly sees that for $0.56<\epsilon<1.61$, ρ takes a nearly constant negative value that differs from zero but is not close to -1, thus indicating that some sort of synchronized motion is established, which, however, is far from complete synchronization. In the following we will show that in this range phase synchronization takes place. In the quasiperiodic regime ($\epsilon>2.4$), the value of the correlation coefficient tends to -1, which means that we have complete synchronization in antiphase.

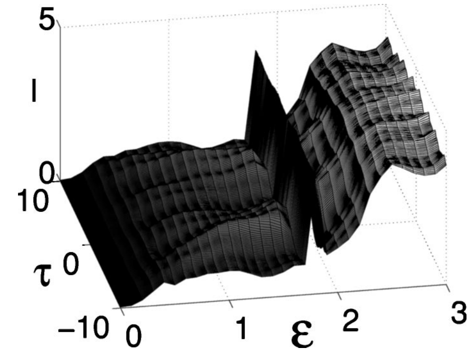

FIG. 3. Average mutual information of the signals $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t+\tau)$ as a function of the time delay τ and the coupling strength ϵ.

The use of linear correlation is subject to caution whenever one deals with chaotic signals. An alternative way to study the synchronization properties of the coupled system is to measure the average mutual information I between the two systems [9]. In the present case, the two bivariate signals are $\mathbf{s}_1(t)=\{x_1(t),y_1(t)\}$ and $\mathbf{s}_{2,\tau}=\mathbf{s}_2(t+\tau)=\{x_2(t+\tau),y_2(t+\tau)\}$, where τ is a delay time (positive or negative), and I is defined by

$$I(\tau)=\sum_{\mathbf{s}_1,\mathbf{s}_{2,\tau}}P_{12}(\mathbf{s}_1,\mathbf{s}_{2,\tau})\log_2\left[\frac{P_{12}(\mathbf{s}_1,\mathbf{s}_{2,\tau})}{P_1(\mathbf{s}_1)P_2(\mathbf{s}_2)}\right], \qquad (2)$$

where P_{12} is the joint probability for measuring $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t+\tau)$ and P_1 and P_2 are the individual probability densities for the measurement of $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively.

Calculations have been performed with "bins" of 15×15 cells in order to construct the histograms of $\mathbf{s}_1$ and $\mathbf{s}_2$, while the joint histogram was composed of a four-dimensional array of 15^4 cells. The results are sketched in Fig. 3 as a function of the two-dimensional parameter space (τ,ϵ). Two plateaus can be distinguished while varying ϵ, the first one occurring for $\epsilon\in[0.3,1.6]$ and the second one for $\epsilon>2.4$. This latter one corresponds to a higher value of I, corroborating the fact that synchronization is an increasing function of ϵ. The structure obtained by varying τ (at fixed ϵ) indicates that there are certain preferred delay times for which the mutual information is maximum. The behavior for ϵ=1.7 is presumably a resonantlike behavior.

In the following we will concentrate our attention on the intermediate coupling regime $0.56<\epsilon<1.61$ and show that it corresponds to a PS state. In particular, we will show that here PS is set in absence of a contracting direction for the phases, at variance with what happens for autonomous chaotic coupled oscillators. Indeed, as it is seen in Fig. 2(a), in this regime the Lyapunov spectrum is composed of one positive, two negative, and two zero exponents. The two vanishing exponents correspond to the time translation invariance (a property which is insensitive to the coupling), and to a coupling induced common phase, respectively.

A practical difficulty in our analysis is that in the range of couplings $0.56<\epsilon<1.61$ (from now on referred to as the PS

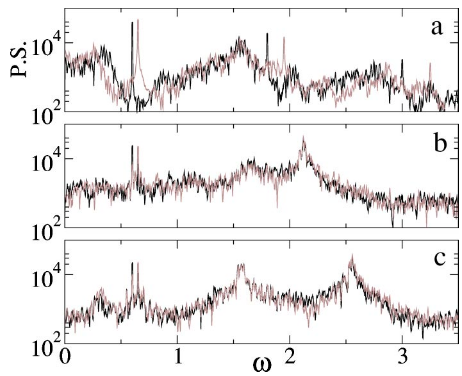

FIG. 4. Power spectrum (in arbitrary units) of the signals x_1 (black) and x_2 (gray) vs the frequency ω for (a) $\epsilon=0$, (b) $\epsilon=1$, and (c) $\epsilon=1.7$.

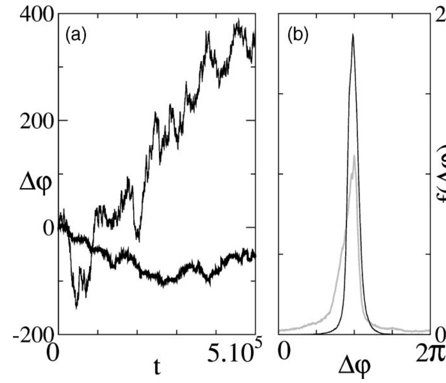

FIG. 5. (a) $\Delta\varphi$ (see text for definition) vs time for $\epsilon=0.4$ (diverging curve) and $\epsilon=1$ (bounded curve), (b) probability distribution function of $\Delta\varphi$ (calculated $mod_{2\pi}$) for $\epsilon=0.4$ (gray curve) and $\epsilon=1$ (black curve).

range), the instantaneous phases of the two oscillators are not easy to calculate, since each oscillator has not a unique center of rotation [see Fig. 1(b)]. A first hint is given by inspection of the power spectra of x_1 and x_2. Figure 4 reports the power spectra of x_1 (black) and x_2 (gray) for $\epsilon=0$ [uncoupled case, (a)], $\epsilon=1$ [in the middle of the PS range, (b)], and $\epsilon=1.7$ [at the border of the PS range, (c)]. In all cases, the Fourier spectra are broad band, consistent with the chaotic dynamics, and they contain two distinct peaks in correspondence with the two external forcing frequencies $\omega_1 = 0.6$ and $\omega_1 = 0.65$. In addition, the uncoupled spectra [Fig. 4(a)] show the presence of two other peaks that are harmonics of the forcing frequencies. As we enter the PS range, the peaks corresponding to the forcing frequencies do not overlap, but a higher peak around $\omega=2.1$ is set common in both spectra, where frequency entrainment is obtained [Fig. 4(b)]. The frequency location of this synchronization peak increases approximately linearly with ϵ. Finally, for $\epsilon=1.7$, Fig. 4(c) shows two "synchronization" peaks at $\omega=1.6$ and $\omega=2.6$. For larger ϵ values, the chaotic attractor becomes structurally unstable. In the region $1.61<\epsilon<2.4$, we have observed a rather rich dynamical behavior where chaotic regions are interrupted by periodic and quasiperiodic windows. As we were primarily interested in the PS and CS regimes we did not investigate further this parameter range and leave it for further studies.

The emergence of a synchronization peak suggests the use of a band-pass filter to properly isolate a filtered signal around the second frequency peak in the Fourier spectra, to which the standard analytic continuation technique [11] can be applied for the evaluation of the instantaneous phase. We emphasize that a unique definition of the phase in a complex system is not available so far, and that a phase would be in any case related to some "band" in the frequency domain. As a consequence, it is often unavoidable to use some band-pass filtering procedure to extract the phase dynamics in a band. Relevant examples where such a procedure has been implemented include brain measurements (like electroencephalograms, magnetoencephalograms, and chaotic laser arrays [3]), where a phase analysis would have not been possible without filtering. In the present case, this is rather clearly motivated from the well-expressed bands in the power spectra. By using Matlab [10], at each ϵ value a pass-band Butterworth filter (order=8, maximum band-pass loss=7 dB, minimum band-stop attenuation=40 dB) has been constructed centered around the second frequency peak in the Fourier spectrum, and the instantaneous phases ϕ_1 and ϕ_2 of the two filtered signals have been evaluated.

Figure 5 reports a long snapshot of the temporal evolution of the maximum instantaneous phase difference between the two fields

$$\Delta\varphi(t)=\phi_1(t)-\phi_2(t) \tag{3}$$

for ϵ values outside ($\epsilon=0.4$) and inside ($\epsilon=1$) the range for which the second Lyapunov exponent vanishes.

Looking at Fig. 5(a), one easily realizes that $\Delta\varphi$ diffuses in an almost random fashion toward infinity at $\epsilon=0.4$. At variance, for $\epsilon=1$, $\Delta\varphi$ behaves alternating long epochs of almost constant value, interrupted by 2π jumps (or phase slips), with no apparent trend. The probability distribution function (PDF) given in Fig. 5(b) confirms that for $\epsilon=1$ the phase difference between the two oscillators is sharply peaked around π. The complete synchronization obtained for larger coupling value ($\epsilon>2.4$) corresponds to antiphase and the PDF is then single peaked around π.

An alternative way of measuring instantaneous phases of chaotic oscillators has been recently proposed [12], based on the frequency locking properties of forced periodic Poincaré oscillators. The method consists in forcing a set of N periodic oscillators by means of a common driving signal x_f, whose frequency and instantaneous phase are unknown. The evolution of the phases of the forced oscillators is ruled by $\dot{\psi}_i = \Omega_i + K x_f \sin(\psi_i)$ $i=1, \ldots, N$, where Ω_i are the natural frequencies of the oscillators, and K is a coupling constant. Due to the coupling, a subset of the N oscillators will phase lock with the external driving, exhibiting an average frequency that can be taken as a measure of the mean frequency of x_s. This is revealed by the emergence of a horizontal plateau [or synchronization plateau (SP)] when plotting the average frequency of the forced oscillators vs their natural frequencies Ω_i. As a consequence, the frequency value of the SP can be taken as a measure of the average frequency of the forcing signal, and the instantaneous phase of x_f can be taken as the phase of any of the oscillators belonging to a SP. We then consider two sets (one for each system) of $N=350$ of such periodic oscillators with natural frequencies distributed between $\Omega_1=0.01$ and $\Omega_{350}=3.5$ with $\Omega_{i+1}-\Omega_i=0.01$, and force them with both signals x_1 and x_2 for $K=0.5$ at various coupling strength ϵ.

At different coupling strengths, the resulting plots reporting the average frequency of the forced oscillators vs their natural frequencies Ω_i contain several different SP's, corresponding to the dominant frequency peaks emerging in the Fourier spectra. It is therefore possible to consider the second SP in these plots (the second peak in the spectra outside the low frequency peaks associated to the two forcing frequencies), and report the mean frequency difference $\Delta\Omega=\langle\omega\rangle_{x_1}-\langle\omega\rangle_{x_2}$, where $\langle\omega\rangle_{x_1}$ ($\langle\omega\rangle_{x_2}$) is the value of this SP when the periodic oscillators are forced with the signal x_1 and x_2, respectively.

The results are presented in Fig. 6, where one notes a frequency synchronization regime ($\Delta\Omega=0$) for coupling strengths ϵ inside the range for which a Lyapunov exponent in the spectrum [see Fig. 2(a)] maintain a value close to zero. This independent check for phase locking allows us to exclude the possibility that the band-pass filtering technique introduced artifacts due to possible side effects.

A still open question concerns the basic dynamical mechanism relating the appearance of phase locking and the properties of the Lyapunov spectrum. In the case of coupled chaotic autonomous oscillators, phase synchronization occurs when a zero Lyapunov exponent becomes negative [5], indicating the emergence of a contractive direction for the phase difference. In such a PS regime, however, the phase difference variable itself is bounded in time, but it does experience nonnegligible residual fluctuations, thus its dynamical behavior is far from being a contractive relaxation toward a constant value [5]. An alternative approach consists in linking the emergence of phase locking to the behavior of the unstable periodic orbits embedded within the chaotic attractor [13]. In the present case, we have chosen the option to analyzing the system almost essentially in terms of the changes in the Lyapunov exponent spectrum. We have observed partial frequency entrainment of dominant peaks in chaotic spectra also in the absence of a negative Lyapunov exponent in the phase difference direction. An open challenge for future work remains to satisfactorily link the features of phase entrainment with the dynamical properties emerging from the measurements of Lyapunov exponents.

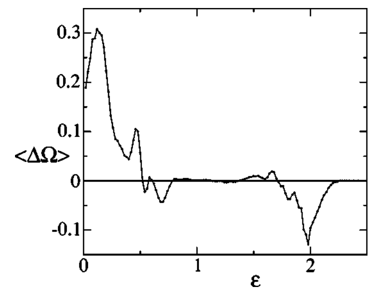

FIG. 6. Average frequency mismatch $\Delta\Omega$ (see text for definition) vs ϵ.

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