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The Pontryagin duality of sequential limits of topological Abelian groups☆

S. Ardanza-Trevijano*, M.J. Chasco
Dept. de Física y Matemática Aplicada, Universidad de Navarra, E-31080 Pamplona, Spain

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Abstract

We prove that direct and inverse limits of sequences of reflexive Abelian groups that are metrizable or k_ω-spaces, but not necessarily locally compact, are reflexive and dual of each other provided some extra conditions are satisfied by the sequences.

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1. Preliminaries

Let **TAG** be the category whose objects are topological Abelian groups and whose morphisms are continuous homomorphisms. Given two objects G and H in **TAG** we denote by $\mathbf{TAG}(G, H)$ the group of morphisms from G to H. Consider the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the Euclidean topology. For any object G in **TAG**, the set of morphisms $\mathbf{TAG}(G, \mathbb{T})$ with the compact open topology is a Hausdorff Abelian group named the character group of G and denoted by $G^\wedge$.

For $f \in \mathbf{TAG}(G, H)$, the adjoint homomorphism $f^\wedge \in \mathbf{TAG}(H^\wedge, G^\wedge)$ is defined by $f^\wedge(\chi) = \chi \circ f$ for $\chi \in H^\wedge$. Thus $(-)^\wedge$ is a contravariant functor from **TAG** to **TAG** (or a covariant functor from **TAG** to $\mathbf{TAG}^{\mathrm{op}}$). There is a natural transformation η from the identity

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* Corresponding author.
E-mail addresses: sardanza@unav.es (S. Ardanza-Trevijano), mjchasco@unav.es (M.J. Chasco).

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functor in **TAG** to the covariant functor $(-)^{\wedge\wedge}$. This can be described by $\eta_G : G \to G^{\wedge\wedge}$ where $[\eta_G(x)](\chi) = \chi(x)$ for any $x \in G$ and $\chi \in G^{\wedge}$.

A group G is said to be *reflexive* if η_G is a topological isomorphism. The celebrated Pontryagin–Van Kampen Theorem states that for **LCA** (the category of locally compact Abelian groups), η is a natural isomorphism, i.e. locally compact Abelian groups are reflexive. The first well-known extension of this Theorem to products and direct sums of reflexive groups was obtained by Kaplan in 1948 [11]. He proved in a subsequent paper the reflexivity of sequential limits (direct and inverse) of **LCA** groups. There have been many studies concerning Pontryagin duality from different perspectives since that time. An excellent survey with almost all representative references about the topic is [8].

Our aim is to extend the results of [12] where Kaplan studied the reflexivity of direct and inverse limits of reduced sequences of **LCA** groups. He obtained the following four isomorphisms:

(i) $(\varinjlim G_n)^{\wedge} \cong \varprojlim G_n^{\wedge}$, (ii) $(\varprojlim G_n)^{\wedge} \cong \varinjlim G_n^{\wedge}$,

(iii) $(\varinjlim G_n)^{\wedge\wedge} \cong \varinjlim G_n$, (iv) $(\varprojlim G_n)^{\wedge\wedge} \cong \varprojlim G_n$.

We study suitable conditions on sequences of Abelian groups (non-necessarily in **LCA**) that ensure the validity of the isomorphisms above. In particular we will obtain the reflexivity of direct or inverse limits for certain families of sequences.

Reflexive topological Abelian groups are Hausdorff. By this reason it is convenient to work in **HTAG**, the category of Hausdorff topological Abelian groups.

Given a directed set $\mathscr{A}$, we can consider it as a category where the objects are the elements $\alpha \in \mathscr{A}$ and the set of morphisms $\mathscr{A}(\alpha, \beta)$ consists of only one element if $\alpha \leqslant \beta$ and is empty otherwise. A *direct system* in **HTAG** is a covariant functor D from a directed set $\mathscr{A}$ to **HTAG**. We use the notation $\{G_\alpha, f_\alpha^\beta, \mathscr{A}\}$ for a direct system, where $G_\alpha = D(\alpha)$ are the groups and $f_\alpha^\beta = D(\mathscr{A}(\alpha, \beta))$ the linking maps.

A *direct limit* or *inductive limit* for a direct system $\{G_\alpha, f_\alpha^\beta, \mathscr{A}\}$ in **HTAG** is a pair $(\varinjlim G_\alpha, \{p_\alpha\}_{\alpha\in\mathscr{A}})$, where $\varinjlim G_\alpha$ is an object in **HTAG** and the p_α's are morphisms in **HTAG**$(G_\alpha, \varinjlim G_\alpha)$ such that $p_\alpha = p_\beta \circ f_\alpha^\beta$ for $\alpha \leqslant \beta$, satisfying the following universal property:

Given an object G' in **HTAG** and morphisms p'_α in **HTAG**(G_α, G') for all $\alpha \in \mathscr{A}$ such that $p'_\alpha = p'_\beta \circ f_\alpha^\beta$ whenever $\alpha \leqslant \beta$, there is a unique morphism p in **HTAG**$(\varinjlim G_\alpha, G')$ such that $p'_\alpha = p \circ p_\alpha$.

The standard construction of the inductive limit in **HTAG** is the following

$$\varinjlim G_\alpha \cong \left(\bigoplus G_\alpha\right) \Big/ \bar{H},$$

where $\bigoplus G_\alpha$ has the final group topology with respect to the inclusions $i_\alpha : G_\alpha \to \bigoplus G_\alpha$ and $\bar{H}$ is the closure of the subgroup H generated by $\{i_\beta \circ f_\alpha^\beta(g_\alpha) - i_\alpha(g_\alpha) : \alpha \leqslant \beta;\ g_\alpha \in G_\alpha\}$.

Dually, an inverse system in **HTAG** is a contravariant functor I from $\mathscr{A}$ to **HTAG** (or equivalently a covariant functor from $\mathscr{A}$ to **HTAG**$^{\mathrm{op}}$, the opposite category). We will denote a generic inverse system by $\{G_\alpha, g^\alpha_\beta, \mathscr{A}\}$ and an *inverse limit* or *projective limit* by a pair $(\varprojlim G_\alpha, \{\pi_\alpha\}_{\alpha\in\mathscr{A}})$, where $\pi_\alpha : \varprojlim G_\alpha \to G_\alpha$.

Given an inverse system $\{G_\alpha, g^\alpha_\beta, \mathscr{A}\}$ in **HTAG**, its inverse limit can be constructed as a subgroup L of the product $\prod G_\alpha$ given by

$$L := \left\{(g_\alpha)_{\alpha\in\mathscr{A}} \in \prod G_\alpha : g^\alpha_\beta(g_\beta) = g_\alpha\right\}.$$

We will call a direct system (resp. inverse system) of topological groups *reduced* if the linking maps f^β_α are injective (resp. if the linking maps g^α_β are onto), and *strict* if the linking maps are embeddings (resp. quotient maps).

We will call *direct sequence* (resp. *inverse sequence*) to any direct system (resp. inverse system) with index set $\mathscr{A} = \mathbb{N}$. In the case of direct sequences the final topology with respect to the inclusions in $\bigoplus G_n$, used in the description of direct limit, coincides with the box topology and with the *asterisk topology* introduced by Kaplan although these topologies do not coincide in general (see [4,11]). The asterisk topology is the appropriate topology for the direct sum in Pontryagin duality. This is the reason to restrict part of the study of duality for direct and inverse limits to the case of sequences.

2. Adjoint functors and Pontryagin duality

We show in this section that certain duality results about limits can be obtained directly from category theory. We use the fact that the contravariant functor $(-)^\wedge$ has a left adjoint in a subcategory of **HTAG** that we proceed to define.

Denote by $\mathbf{HTAG}_\eta$ the full subcategory of **HTAG** of all Hausdorff Abelian topological groups G for which $\{\eta_G, \eta_{G^\wedge}, \eta_{G^{\wedge\wedge}}, \ldots\}$ are continuous homomorphisms. For each pair G and H of objects in $\mathbf{HTAG}_\eta$ and morphism $f : G \to H^\wedge$, there is a unique morphism $f' : H \to G^\wedge$ such that $f = (f')^\wedge \circ \eta_G$. In fact, for $h \in H$ and $g \in G$, $f'(h)(g) = f(g)(h)$ and the map $F : \mathbf{HTAG}_\eta(G, H^\wedge) \to \mathbf{HTAG}_\eta(H, G^\wedge)$ which maps f to f' is continuous. Hence, the functor $(-)^\wedge : \mathbf{HTAG}_\eta^{\mathrm{op}} \to \mathbf{HTAG}_\eta$ is adjoint to $(-)^\wedge : \mathbf{HTAG}_\eta \to \mathbf{HTAG}_\eta^{\mathrm{op}}$. In particular, the contravariant functor $(-)^\wedge : \mathbf{HTAG}_\eta \to \mathbf{HTAG}_\eta$ transforms direct into inverse limits whenever they exist [10, p. 307].

We are going to consider two full subcategories of $\mathbf{HTAG}_\eta$ related by the $(-)^\wedge$ functor: the category **M** whose objects are metrizable and the category $\mathbf{K}_\omega$ whose objects are k_ω-spaces.

Recall that a Hausdorff topological space X is a *k-space* if any subset O of X is open whenever $O \cap K$ is open in K for any K compact. The space X is *hemicompact* if $X = \bigcup_{n\in\mathbb{N}} K_n$, where $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact subsets and every compact subset of X is contained in one of the K_n's. In the literature hemicompact k-spaces are also called *k_ω-spaces*; they were introduced as a generalization of countable CW-complexes (see [9]). Another way to introduce a k_ω-space is to say that there exists an increasing sequence of compact Hausdorff subsets K_n covering X, with $K_n \subset K_{n+1}$, and such that X has the weak topology with respect to the groups K_n, that is, F is closed in X if and only if $F \cap K_n$ is closed in K_n for each n.

It is well known that if an object G of **HTAG** is a k-space, then η_G is continuous (see [13]). Metrizable and k_ω-spaces are obviously k-spaces. Since the character group of a metrizable topological Abelian group is a k_ω-space (see [1] or [6]) and the character group of a hemicompact group is again metrizable, we obtain that the categories **M** and $\mathbf{K}_\omega$ are full subcategories of $\mathbf{HTAG}_\eta$ which are duals of each other.

Proposition 2.1. *The category* $\mathbf{K}_\omega$ *has sequential direct limits. Dually, the category* **M** *has sequential inverse limits.*

Proof. We only need to prove that $\bigoplus G_n$, with the final group topology, is a k_ω-space, since the image by a quotient mapping of a k_ω-space onto a Hausdorff space is a k_ω-space.

Since the G_n's are k_ω-spaces there exists an increasing sequence of compact Hausdorff subsets $(K_n^j)_{j\in\mathbb{N}}$ containing e_{G_n} covering each G_n such that F is closed in G_n if and only if $F \cap K_n^j$ is closed in K_n^j, for each j. The sequence

$$(C_N)_{N\in\mathbb{N}} = (i_1(K_1^N) + i_2(K_2^N) + \cdots + i_N(K_N^N))_N$$

is increasing and covers $\bigoplus G_n$. Take now $B \subset \bigoplus G_n$ such that $B \cap C_N$ is closed in C_N for all $N \in \mathbb{N}$. We need to show that B is closed in $\bigoplus G_n$. Since $\bigoplus G_n$ has been endowed with the final topology with respect to the inclusions $i_n : G_n \to \bigoplus G_n$, it is enough to prove that $i_n^{-1}(B)$ is closed in G_n for all $n \in \mathbb{N}$, which occurs if and only if $i_n^{-1}(B) \cap K_n^N$ is closed in K_n^N for all $N \in \mathbb{N}$. Fix n and N natural numbers. If $n \geqslant N$ then $i_n(K_n^N) \subset i_n(K_n^n) \subset C_n$; otherwise $i_n(K_n^N) \subset C_N$. It now follows that $B \cap i_n(K_n^N)$ is closed in $i_n(K_n^N)$, and therefore $i_n^{-1}(B) \cap K_n^N = i_n^{-1}(B \cap i_n(K_n^N))$ is closed in K_n^N as we wanted to show.

Since a subgroup of a countable product of metrizable groups is metrizable, we obtain that the inverse limit of a sequence of metrizable groups is also metrizable. $\square$

Corollary 2.2. *Let* $\varinjlim G_n$ *be a limit of Abelian topological groups which are also* k_ω-*spaces then*

$$\left(\varinjlim G_n\right)^\wedge \cong \varprojlim G_n^\wedge.$$

Proof. This isomorphism is a direct consequence of the previously commented fact that $(-)^\wedge$ has a left adjoint and the existence of the required direct and inverse limits. $\square$

We have obtained the first of the isomorphisms announced in the introduction. The other three cannot be achieved just by categorical arguments and require specific machinery.

3. Reflexivity of the inverse limit

This section is devoted show how the reflexivity of the inverse limit of sequences in **M** follows from the nice properties that $\varprojlim G_\alpha$ has as a subgroup of the product $\prod G_\alpha$.

Let G be an object of **HTAG**. A subgroup H of G is called *dually embedded* if every character of H extends to a character of G, i.e. $i^\wedge : G^\wedge \to H^\wedge$ is onto. A subgroup H of

G is called *dually closed* in G if for every $x \in G \backslash H$ there exists a character $\chi \in G^{\wedge}$ with $\chi(H) = e_{\mathbb{T}}$ and $\chi(x) \neq e_{\mathbb{T}}$. It is easy to check that a dually closed subgroup H is a closed subgroup of G such that $\eta_{G/H}$ is injective.

Proposition 3.1. *Let $\{G_\alpha, g_\beta^\alpha; \mathscr{A}\}$ be an inverse system of Hausdorff Abelian groups and $\varprojlim G_\alpha$ its inverse limit, then*:

1. *If the inverse system is reduced, $\varprojlim G_\alpha$ is dually embedded in $\prod_{\alpha \in \mathscr{A}} G_\alpha$.*
2. *If all η_{G_α} are injective, $\varprojlim G_\alpha$ is dually closed in $\prod_{\alpha \in \mathscr{A}} G_\alpha$.*

Proof. Denote $\mathbb{T}_+ = \{z \in \mathbb{T} : \operatorname{Re} z > 0\}$ and let $\varphi : \varprojlim G_\alpha \to \mathbb{T}$ be a continuous homomorphism. We are going to extend φ to $G = \prod_{\alpha \in \mathscr{A}} G_\alpha$.

Since φ is continuous $\varphi^{-1}(\mathbb{T}_+)$ is a neighborhood of the neutral element $e_{\varprojlim G_\alpha}$ and contains an element $\varprojlim G_\alpha \cap U$, where $U = U_{\alpha_1} \times \cdots \times U_{\alpha_k} \times \prod_{\substack{\alpha \neq \alpha_i \\ i=1,\dots,k}} G_\alpha$, U_{α_i} is a neighborhood of the identity in G_{α_i} and $\alpha_k > \alpha_i$ for $i \neq k$. Denote $U_1 = U_{\alpha_1} \times \cdots \times U_{\alpha_k}$, $G_1 = G_{\alpha_1} \times \cdots \times G_{\alpha_k}$, $G_2 = \prod_{\substack{\alpha \neq \alpha_i \\ i=1,\dots,k}} G_\alpha$ and let $\pi_1 : G \to G_1$ the natural projection. Notice that $\varphi(\varprojlim G_\alpha \cap G_2)$ is the trivial subgroup since $\varphi(\varprojlim G_\alpha \cap G_2) \subset \varphi(\varprojlim G_\alpha \cap U) \subset \mathbb{T}_+$. Hence we can define $\varphi_1 : \pi_1(\varprojlim G_\alpha) \to \mathbb{T}$ as $\varphi_1(\pi_1((x_\alpha)_{\alpha \in \mathscr{A}})) = \varphi((x_\alpha)_{\alpha \in \mathscr{A}})$.

The morphism φ_1 is well defined: if $\pi_1((x_\alpha)_{\alpha \in \mathscr{A}}) = \pi_1((x'_\alpha)_{\alpha \in \mathscr{A}})$ then $(x_\alpha)_{\alpha \in \mathscr{A}} - (x'_\alpha)_{\alpha \in \mathscr{A}} \in \varprojlim G_\alpha \cap G_2$. Now $\varphi((x_\alpha)_{\alpha \in \mathscr{A}} - (x'_\alpha)_{\alpha \in \mathscr{A}}) = e_{G_\alpha}$, hence $\varphi((x_\alpha)_{\alpha \in \mathscr{A}}) = \varphi((x'_\alpha)_{\alpha \in \mathscr{A}})$.

Let us prove now that φ_1 is continuous: Since $\varphi^{-1}(\mathbb{T}_+) \supset \varprojlim G_\alpha \cap U$, we have that $\varphi_1^{-1}(\mathbb{T}_+) \supset \pi_1(\varprojlim G_\alpha \cap U) = \pi_1(\varprojlim G_\alpha) \cap U_1$ and $\pi_1(\varprojlim G_\alpha) \cap U_1$ is a neighborhood of the neutral element in $\pi_1(\varprojlim G_\alpha)$.

Since the g_β^α's are surjective we can write:

$$\pi_1\left(\varprojlim G_\alpha\right) = \{(g_{\alpha_k}^{\alpha_1}(x_{\alpha_k}), \dots, g_{\alpha_k}^{\alpha_{k-1}}(x_{\alpha_k}), x_{\alpha_k}) : x_{\alpha_k} \in G_{\alpha_k}\}.$$

This allows us to define another homomorphism $\varphi_{\alpha_k} : G_{\alpha_k} \to \mathbb{T}$ such that $\varphi_{\alpha_k}(x_{\alpha_k}) = \varphi_1(g_{\alpha_k}^{\alpha_1}(x_{\alpha_k}), \dots, g_{\alpha_k}^{\alpha_{k-1}}(x_{\alpha_k}), x_{\alpha_k})$. In fact, we can obtain φ_{α_k} as the composition $\varphi_1 \circ i$, where $i : G_{\alpha_k} \to \pi_1(\varprojlim G_\alpha)$, is the continuous homomorphism defined by $i(x_{\alpha_k}) = (g_{\alpha_k}^{\alpha_1}(x_{\alpha_k}), \dots, g_{\alpha_k}^{\alpha_{k-1}}(x_{\alpha_k}), x_{\alpha_k})$.

Hence φ_{α_k} is a continuous homomorphism. If we take $\widetilde{\varphi} = \varphi_{\alpha_k} \circ \pi_{\alpha_k}$ we have a continuous homomorphism that extends φ. If $(x_\alpha)_{\alpha \in \mathscr{A}} \in \varprojlim G_\alpha$ then $g_\beta^\alpha(x_\beta) = x_\alpha$ for each $\beta \geqslant \alpha$.

Now it follows that $\widetilde{\varphi}((x_\alpha)_{\alpha \in \mathscr{A}}) = \varphi_{\alpha_k}(x_{\alpha_k}) \in G_{\alpha_k}$ and $\varphi_{\alpha_k}(x_{\alpha_k}) = \varphi_1(g_{\alpha_k}^{\alpha_1}(x_{\alpha_k}), \dots, g_{\alpha_k}^{\alpha_{k-1}}(x_{\alpha_k}), x_{\alpha_k}) = \varphi_1(\pi_1((x_\alpha)_{\alpha \in \mathscr{A}})) = \varphi((x_\alpha)_{\alpha \in \mathscr{A}})$. This proves the first part.

The second part is Lemma 5.28 in [1]. □

Theorem 3.2. *Let $\{G_n, g_m^n, n \leqslant m\}$ a reduced inverse sequence of metrizable, reflexive, topological groups. Then $\varprojlim G_n$ is a metrizable, reflexive topological group.*

Proof. We are under the hypothesis of Proposition 3.1, hence $\varprojlim G_n$ is dually closed and dually embedded in $\prod G_n$.

Each G_n is metrizable and reflexive, hence we have that $\prod G_n$ is metrizable and reflexive (see [11]). Every dually closed and dually embedded subgroup of a metrizable reflexive group is reflexive [7], hence $\varprojlim G_n$ is reflexive. $\square$

4. Limits and local quasi-convexity

In this section we introduce the category $\mathbf{HTAG}_{\mathrm{lqc}}$ of locally quasi-convex Hausdorff Abelian groups. Inverse limits of sequences of reflexive groups in this new category coincide with the ones in **HTAG**. We will prove that, under certain restrictions on the linking maps, the direct limit (in $\mathbf{HTAG}_{\mathrm{lqc}}$) of sequences of reflexive k_ω-objects exists and is reflexive.

For a Hausdorff topological group G the *polar* of a subset $A \subset G$ is the set $A^{\triangleright} = \{\chi \in G^\wedge : \chi(A) \subset \mathbb{T}_+\}$ where $\mathbb{T}_+ = \{z \in \mathbb{T} : \mathrm{Re}\, z > 0\}$. The *inverse* polar of a set $B \subset G^\wedge$ is the set $B^{\triangleleft} = \{x \in G : \chi(x) \in \mathbb{T}_+, \ \forall \chi \in B\}$.

A subset A of a topological group is said to be *quasi-convex* if $A^{\triangleright\triangleleft} = A$ i.e. if for every $x \in G \backslash A$ there exists $\chi \in G^\wedge$ such that $\chi(A) \in \mathbb{T}_+$ but $\chi(x) \notin \mathbb{T}_+$.

A topological Abelian group is *locally quasi-convex* if it has a basis of neighborhoods of the neutral element formed by quasi-convex sets. In particular, the character group $G^\wedge$ of any topological group G is always locally quasi-convex, hence reflexive groups are locally quasi-convex.

Every Hausdorff Abelian topological group, can be obtained as a quotient of a locally quasi-convex group [1, p. 61]. This implies that local quasi-convexity is not preserved in general by quotients and as a consequence, the direct limit of locally quasi-convex groups in **HTAG** is not always locally quasi-convex. The problem can be avoided if we restrict to the category $\mathbf{HTAG}_{\mathrm{lqc}}$, of locally quasi-convex Hausdorff Abelian groups.

For any topological Hausdorff Abelian group (G, τ) with $\{U_i\}$ as neighborhood basis at e_G, we define the *associated locally quasi-convex topology*, τ_{lqc}, on G taking $\{U_i^{\triangleright\triangleleft}\}$ as neighborhood basis of e_G. It is the finest topology contained in τ, such that (G, τ_{lqc}) is a (non-necessarily Hausdorff) locally quasi-convex group [5]. The correspondence $G \mapsto (G, \tau_{\mathrm{lqc}})/\overline{\{e_G\}}$ defines a functor $\mathscr{LQC} : \mathbf{HTAG} \to \mathbf{HTAG}_{\mathrm{lqc}}$.

Remark. The group (G, τ_{lqc}) is Hausdorff if and only if η_G is injective [1, p. 35]. For such a group $\mathscr{LQC}(G) = (G, \tau_{\mathrm{lqc}})$.

Let $\{(G_n)_{n\in\mathbb{N}}, f_n^m\}$ be a direct sequence of locally quasi-convex Abelian topological groups. We will see that for this kind of sequences, we can construct a direct limit in the category $\mathbf{HTAG}_{\mathrm{lqc}}$. For this purpose we need an auxiliary result.

Lemma 4.1. *Given a reduced inverse system $\{G_n, g_m^n; \mathbb{N}\}$ of Hausdorff topological groups, the polar of $\varprojlim G_n$ is the subgroup of $\bigoplus G_n^\wedge$ generated by $\{i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n) : n \leqslant m, \varphi_n \in G_n^\wedge\}$. We will denote this subgroup by $\langle i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n)\rangle$.*

Proof. Let us first show that

$$i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n) \in \left(\varprojlim G_n\right)^{\triangleright}.$$

If $(x_n)_{n\in\mathbb{N}} \in \varprojlim G_n$, we have that $g_m^n(x_m) = x_n$, hence

$$\begin{aligned}(i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n))((x_n)_{n\in\mathbb{N}}) &= \varphi_n(x_n) - (g_m^n)^\wedge(\varphi_n)(x_m)\\ &= \varphi_n(x_n) - \varphi_n(g_m^n(x_m)) = \varphi_n(x_n) - \varphi_n(x_n) = e_{\mathbb{T}},\end{aligned}$$

and we have proven one inclusion.

We are left to prove that $\langle i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n)\rangle \supset (\varprojlim G_n)^{\triangleright}$.

Any element $(\varphi_n)_{n\in\mathbb{N}} \in (\varprojlim G_n)^{\triangleright}$ can be represented as a finite sum

$$(\varphi_n)_{n\in\mathbb{N}} = i_{n_1}(\varphi_{n_1}) + \cdots + i_{n_k}(\varphi_{n_k}).$$

Consider now an arbitrary element $x_{n_k} \in G_{n_k}$ and let $(x_n)_{n\in\mathbb{N}}$ an element of the inverse limit with n_k coordinate x_{n_k}. We know that $g_m^n(x_m) = x_n$, $n \leqslant m$ and since $(\varphi_n)_{n\in\mathbb{N}}$ is in the polar of $\varprojlim G_n$, we have

$$(\varphi_n)_{n\in\mathbb{N}}((x_n)_{n\in\mathbb{N}}) = e_{\mathbb{T}}.$$

We can use both facts together to obtain:

$$\begin{aligned}(\varphi_n)_{n\in\mathbb{N}}((x_n)_{n\in\mathbb{N}}) &= \varphi_{n_1}(x_{n_1}) + \cdots + \varphi_{n_k}(x_{n_k})\\ &= (\varphi_{n_1} g_{n_k}^{n_1} + \cdots + \varphi_{n_{k-1}} g_{n_k}^{n_{k-1}} + \varphi_{n_k})(x_{n_k})\\ &= ((g_{n_k}^{n_1})^\wedge(\varphi_{n_1}) + \cdots + (g_{n_k}^{n_{k-1}})^\wedge(\varphi_{n_{k-1}}) + \varphi_{n_k})(x_{n_k})\\ &= e_{\mathbb{T}}\end{aligned}$$

and hence

$$((g_{n_k}^{n_1})^\wedge(\varphi_{n_1}) + \cdots + (g_{n_k}^{n_{k-1}})^\wedge(\varphi_{n_{k-1}}) + \varphi_{n_k}) = e_{G_{n_k}^\wedge}.$$

We can now subtract this term from the expression of $(\varphi_n)_{n\in\mathbb{N}}$ which is enough to obtain our result. More concretely,

$$\begin{aligned}(\varphi_n)_{n\in\mathbb{N}} &= i_{n_1}(\varphi_{n_1}) + \cdots + i_{n_k}(\varphi_{n_k})\\ &= i_{n_1}(\varphi_{n_1}) + \cdots + i_{n_k}(\varphi_{n_k})\\ &\quad - i_{n_k}((g_{n_k}^{n_1})^\wedge(\varphi_{n_1}) + \cdots + (g_{n_k}^{n_{k-1}})^\wedge(\varphi_{n_{k-1}}) + \varphi_{n_k})\\ &= i_{n_1}(\varphi_{n_1}) - i_{n_k}(g_{n_k}^{n_1})^\wedge(\varphi_{n_1})\\ &\quad + \cdots + i_{n_{k-1}}(\varphi_{n_{k-1}}) - i_{n_k}(g_{n_k}^{n_{k-1}})^\wedge(\varphi_{n_{k-1}}) + i_{n_k}(\varphi_{n_k}) - i_{n_k}(\varphi_{n_k}),\end{aligned}$$

from which we conclude $(\varprojlim G_n)^{\triangleright} \subset \langle i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n)\rangle$. $\square$

Recall that for a direct sequence $\{(G_n)_{n\in\mathbb{N}}, f_n^m : G_n \to G_m,\ n \leqslant m\}$ the limit in **HTAG** is $(\bigoplus G_n)/\overline{H}$ where H is the subgroup of $\bigoplus_{n\in\mathbb{N}} G_n$ generated by $\{i_m \circ f_n^m(g) - i_n(g) : g \in G_n\}$ with $i_n : G_n \to \bigoplus G_n$ the canonical inclusions.

For reflexive groups we obtain the following proposition.

Proposition 4.2. *Given a direct sequence of reflexive topological groups the direct limit in* $\mathbf{HTAG}_{\mathrm{lqc}}$ *is*

$$\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n \cong \left(\left(\bigoplus G_n\right) \Big/ \bar{H}, \tau_{\mathrm{lqc}}\right),$$

where τ_{lqc} *is the associated locally quasi-convex topology*.

Proof. The functor $\mathscr{LQC} : \mathbf{HTAG} \to \mathbf{HTAG}_{\mathrm{lqc}}$ defined on the objects by $\mathscr{LQC}(G) = (G, \tau_{\mathrm{lqc}})/\overline{\{e_G\}}$, is a left adjoint to the inclusion $\mathbf{HTAG}_{\mathrm{lqc}} \to \mathbf{HTAG}$. Hence it preserves direct limits, in particular, $\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n \cong \mathscr{LQC}(\varinjlim_{\mathbf{HTAG}} G_n)$. □

Remark. In order to obtain the direct limit for sequences of reflexive topological groups in the categories **HTAG** and $\mathbf{HTAG}_{\mathrm{lqc}}$, we have used two topologies on $(\bigoplus G_n)/\bar{H}$. In **HTAG** we have taken the quotient group topology denoted by τ. In $\mathbf{HTAG}_{\mathrm{lqc}}$ we have considered τ_{lqc}, which is the associated locally quasi-convex topology to τ. We note here that the respective algebraic and topological duals are the same whenever the groups are k_ω-spaces, since $((\bigoplus G_n)/\bar{H}, \tau)^\wedge$ is a k-space with the same compact sets as $((\bigoplus G_n)/\bar{H}, \tau_{\mathrm{lqc}})^\wedge$. This implies that the isomorphism obtained in Corollary 2.2 is valid when the direct limit is taken in any of the categories **HTAG** or $\mathbf{HTAG}_{\mathrm{lqc}}$. Note also that for a sequence $\{G_n\}_{n\in\mathbb{N}}$ of reflexive groups, $\varprojlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n = \varprojlim_{\mathbf{HTAG}} G_n$ since products and subgroups of locally quasi-convex groups are locally quasi-convex. Hence we will denote both limits by $\varprojlim G_n$.

Theorem 4.3. *Let* $\{G_n, g_m^n, n \leqslant m\}$ *a reduced inverse sequence of metrizable, reflexive, topological groups. Then*

$$\left(\varprojlim G_n\right)^\wedge \cong \varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^\wedge.$$

Proof. We have from Lemma 4.1

$$\begin{aligned}\left(\varprojlim G_n\right)^\triangleright &= \left\{(\varphi_n)_{n\in\mathbb{N}} \in \bigoplus G_n^\wedge : \sum \varphi_n(x_n) = e_{\mathbb{T}}, \text{ for all } (x_n) \in \varprojlim G_n\right\}\\ &= \langle i_n(\varphi_n) - i_m(g_m^n)^\wedge(\varphi_n), : n \leqslant m, \varphi_n \in G_n^\wedge\rangle.\end{aligned}$$

Considering the locally quasi-convex topology τ_{lqc}, associated to the quotient topology in $(\bigoplus G_n^\wedge)/(\varprojlim G_n)^\triangleright$ we have an object isomorphic to $\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^\wedge$ in $\mathbf{HTAG}_{\mathrm{lqc}}$. We still need to prove that $(\varprojlim G_n)^\wedge$ is topologically isomorphic to this object.

We have proven before that $\varprojlim G_n$ is dually closed and embedded in $\prod G_n$. Let $i: \varprojlim G_n \to \prod G_n$ be the inclusion. Since $\varprojlim G_n$ is dually embedded, the continuous homomorphism $i^\wedge : (\prod G_n)^\wedge \to (\varprojlim G_n)^\wedge$ is onto and has $(\varprojlim G_n)^\triangleright$ as kernel.

Therefore we have a continuous isomorphism

$$\psi\colon \left(\left(\prod G_n\right)^{\wedge} \Big/ \left(\varprojlim G_n\right)^{\triangleright}, \tau_{\mathrm{lqc}}\right) \to \left(\varprojlim G_n\right)^{\wedge},$$

with both $((\prod G_n)^{\wedge}/(\varprojlim G_n)^{\triangleright}, \tau_{\mathrm{lqc}})$ and $(\varprojlim G_n)^{\wedge}$ locally quasi-convex. We obtain from Lemma 14.8 in [3] that ψ is open and hence a topological isomorphism. □

Theorem 4.4. *If* $\{G_n, g_m^n\}$ *is a reduced inverse sequence of reflexive metrizable groups, its dual* $\{G_n^{\wedge}, (g_m^n)^{\wedge}\}$ *is a reduced direct sequence of reflexive* k_ω*-spaces and* $\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^{\wedge}$ *is reflexive.*

Proof. Every reflexive metrizable group G is complete, its dual group $G^{\wedge}$ is a k_ω-space and its bidual group $G^{\wedge\wedge}$ is again metrizable (see [6]).

Each g_m^n is an onto map, hence its dual $(g_m^n)^{\wedge}$ is injective and therefore $\{G_n^{\wedge}, (g_m^n)^{\wedge}\}$ is a reduced direct sequence of k_ω-space groups.

Now we see that $\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^{\wedge}$ is reflexive.

Since $(\varinjlim_{\mathbf{HTAG}} G_n)^{\wedge} \cong (\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n)^{\wedge}$, from Corollary 2.2 and the reflexivity of the G_n's we obtain

$$(\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^{\wedge})^{\wedge\wedge} = ((\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^{\wedge})^{\wedge})^{\wedge} \cong \left(\varprojlim G_n^{\wedge\wedge}\right)^{\wedge} \cong \left(\varprojlim G_n\right)^{\wedge}$$

and since $\{G_n, (g_m^n)\}$ is a reduced inverse sequence of metrizable groups, Theorem 4.3 gives $(\varprojlim G_n)^{\wedge} \cong \varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^{\wedge}$ which completes the proof. □

Theorem 4.5. *If* $\{G_n, f_n^m\}$ *is a strict direct sequence of reflexive groups in* $\mathbf{K}_\omega$ *such that* $f_n^m(G_n)$ *is dually embedded in* G_m *for all* $n \leqslant m$*, then* $\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n$ *is reflexive.*

Proof. Since $f_n^m(G_n)$ is dually embedded in G_m, we have that $(f_n^m)^{\wedge}\colon G_m^{\wedge} \to G_n^{\wedge}$ is onto. Therefore $\{G_n^{\wedge}, (f_n^m)^{\wedge}\}$ is a reduced inverse sequence of reflexive metrizable groups. Again the fact that $(\varinjlim_{\mathbf{HTAG}} G_n)^{\wedge} \cong (\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n)^{\wedge}$ together with Corollary 2.2 yield $(\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n)^{\wedge\wedge} \cong (\varprojlim G_n^{\wedge})^{\wedge}$. Now Theorem 4.3 and the reflexivity of the G_n's give $(\varprojlim G_n^{\wedge})^{\wedge} \cong \varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n^{\wedge\wedge} \cong \varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}} G_n$. □

We conclude this article with an application of Theorems 3.2 and 4.5 to nuclear groups.

The class of nuclear groups contains all locally compact Abelian groups and the additive groups underlying nuclear vector spaces. Moreover, it is closed with respect to the operations of taking subgroups, Hausdorff quotients, arbitrary products and countable direct sums. The intensive study of nuclear groups developed since their introduction has produced important results (see [2] for a survey). Some of these results are related with strong reflexivity. A reflexive group G is *strongly reflexive* if every closed subgroup and every Hausdorff quotient group of G and of $G^\wedge$ is reflexive. It follows that locally compact groups are strongly reflexive. This notion was defined by Brown, Higgins and Morris in [4] where they showed that countable products and sums of lines and circles are strongly reflexive. Concerning nuclear groups, Banaszczyk proved the strong reflexivity of metrizable complete nuclear groups [3, p. 153]. In particular, countable products of metrizable reflexive nuclear groups are strongly reflexive. Note that strong reflexivity is not a productive property [3, p. 155]. We obtain the following dual result for direct sums.

Corollary 4.6. *Let* $(G_n)_{n\in\mathbb{N}}$ *a collection of strongly reflexive,* k_ω*-spaces, nuclear Abelian topological groups. Then its countable sum* $\bigoplus_{n\in\mathbb{N}} G_n$ *is strongly reflexive.*

Proof. Note first that $H=\bigoplus G_n$, is reflexive.

Let Q be a closed arbitrary subgroup of H. For each $n\in\mathbb{N}$, $H_n=G_1+\cdots+G_n$ is a subgroup of H. Consider $Q_n=Q\cap H_n$. We can now define bonding maps for all $n\leqslant m$:

$$f_n^m\colon H_n\to H_m,$$
$$(f_n^m)'\colon Q_n\to Q_m,$$
$$(f_n^m)''\colon H_n/Q_n\to H_m/Q_m,$$

which are all embeddings between nuclear groups. Then we obtain $Q\cong\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}}\{Q_n,(f_n^m)'\}$ and $H/Q\cong\varinjlim_{\mathbf{HTAG}_{\mathrm{lqc}}}\{H_n/Q_n,(f_n^m)''\}$, which are direct limits of strict sequences of reflexive nuclear groups that are k_ω-spaces. They are all subgroups of nuclear groups and hence dually embedded [3, p. 82]. We can now apply Theorem 4.5 which ensures that the limits Q and H/Q are reflexive.

Consider now $L=\prod G_n^\wedge$, with $G_n^\wedge$ reflexive, metrizable, then L is reflexive. Let P be an arbitrary closed subgroup of L and $L_n=G_1^\wedge\times\cdots\times G_n^\wedge$ with $\pi_n\colon L\to L_n$ the canonical projection. Define $P_n=\overline{\pi_n(P)}$, a closed subgroup of L_n, and bonding maps for each $n\leqslant m$:

$$g_m^n\colon L_m\to L_n,$$
$$(g_m^n)'\colon P_m\to P_n,$$
$$(g_m^n)''\colon L_m/P_m\to L_n/P_n.$$

Now we have the following isomorphisms: $P\cong\varprojlim\{P_n,(g_m^n)'\}$ and $L/P\cong\varprojlim\{L_n/P_n,(g_m^n)''\}$ which are inverse limits of sequences of metrizable topological groups with onto bonding maps. Hence P and L/P are reflexive by Theorem 3.2. □

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