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Topology and its Applications 154 (2007) 741–748
Topology and its Applications
www.elsevier.com/locate/topol

A class of angelic sequential non-Fréchet–Urysohn topological groups

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Received 22 July 2006; accepted 28 August 2006

Abstract

Fréchet–Urysohn (briefly F-U) property for topological spaces is known to be highly non-multiplicative; for instance, the square of a compact F-U space is not in general Fréchet–Urysohn [P. Simon, A compact Fréchet space whose square is not Fréchet, Comment. Math. Univ. Carolin. 21 (1980) 749–753. [27]]. Van Douwen proved that the product of a metrizable space by a Fréchet–Urysohn space may not be (even) sequential. If the second factor is a topological group this behaviour improves significantly: we have obtained (Theorem 1.6(c)) that the product of a first countable space by a F-U topological group is a F-U space. We draw some important consequences by interacting this fact with Pontryagin duality theory. The main results are the following:

(1) If the dual group of a metrizable Abelian group is F-U, then it must be metrizable and locally compact.
(2) Leaning on (1) we point out a big class of hemicompact sequential non-Fréchet–Urysohn groups, namely: the dual groups of metrizable separable locally quasi-convex non-locally precompact groups. The members of this class are furthermore complete, strictly angelic and locally quasi-convex.
(3) Similar results are also obtained in the framework of locally convex spaces.

Another class of sequential non-Fréchet–Urysohn complete topological Abelian groups very different from ours is given in [E.G. Zelenyuk, I.V. Protasov, Topologies of Abelian groups, Math. USSR Izv. 37 (2) (1991) 445–460. [32]].

MSC: primary 22A05; secondary 46A16

Keywords: Abelian topological group; Compact-open topology; Fréchet–Urysohn; Sequential space; k-space; Locally convex space

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[1] The three authors were partially supported by DGICYT BFM 2003-05878, and FEDER funds.
[2] Supported by Proyectos de Investigación Santander/Complutense PR 27/05-14039.

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doi:10.1016/j.topol.2006.08.008

0. Preliminaries and notation

This short note was originated by the following question of D. Dikranjan: if the compact subsets of a topological space X are Fréchet–Urysohn must the k-extension of X be Fréchet–Urysohn? Together with the negative answer to this question, the results obtained here make us conclude, loosely speaking, that very good convergence properties for the compact subsets of a *sequential* hemicompact topological group are not shared by all the closed subsets. More precisely, let $\mathcal{C}$ be the class of all Abelian topological groups which are metrizable, separable, dually separated, with non-locally compact dual group. The class $\mathcal{C}^\wedge$ of the dual groups of members of $\mathcal{C}$, is a family of angelic, sequential, hemicompact complete topological groups which are not Fréchet–Urysohn, while their compact subsets are even metrizable. In the same spirit, if $\mathcal{V}$ denotes the class formed by the infinite dimensional topological vector spaces which are metrizable, separable, dually separated, and $\mathcal{V}^*$ the class formed by the dual vector spaces of members of $\mathcal{V}$ endowed with the compact-open topology, then $\mathcal{V}^*$ is a class of sequential hemicompact non-Fréchet–Urysohn complete locally convex topological vector spaces.

Now some notation needed: Denote by $\mathbb{K}$ either the real or the complex field and by E a topological vector space over $\mathbb{K}$. The vector space of all continuous linear functionals on E will be denoted by E^*, and E^*_c and E^*_β will stand for the same set endowed with the compact-open topology or with the topology of uniform convergence on the bounded subsets of E respectively. The space E^*_β is usually called the strong dual of E, and we have called E^*_c the Pontryagin dual of E. The polar of a subset $U \subset E$ will be denoted by $U^\circ := \{f \in E^*;\ |f(x)| \leqslant 1,\ \forall x \in U\}$.

Next we give the analogue setting for Abelian topological groups. From now on all the groups considered must be Abelian, even if it is not explicitly said. Let us denote by G a topological Abelian group and by $G^\wedge$ the group of all continuous homomorphisms from G into $\mathbb{T}$, where $\mathbb{T}$ is the unit circle in the complex plane, with multiplicative structure. If $G^\wedge$ is endowed with the compact-open topology we will write $G^\wedge_c$, and this is called in the literature the Pontryagin dual—or just the dual—of G. The polar of a subset $U \subset G$ is defined by $U^\triangleright := \{\chi \in G^\wedge\colon\ \chi(x) \in \mathbb{T}_+,\ \forall x \in U\}$, where $\mathbb{T}_+ = \{x \in \mathbb{T};\ \operatorname{Re} x \geqslant 0\}$. Also, the inverse polar of a subset $V \subset G^\wedge$ is defined by $V^\triangleleft := \{x \in G\colon\ \chi(x) \in \mathbb{T}_+,\ \forall \chi \in V\}$ and a set $U \subset G$ is called quasi-convex if $U = (U^\triangleright)^\triangleleft$. The group G is locally quasi-convex if its neutral element has a basis of neighborhoods consisting of quasi-convex sets.

1. Auxiliary results

For the reader's convenience we enter the forest of sequentiality and state the definitions and results to be used in the main theorems.

Definitions. Let X be a topological space.

- A subset $M \subset X$ is said to be *sequentially closed* if from $(x_n)_{n\in\mathbb{N}} \subset M$, $x \in X$ and $\lim_n x_n = x$, it follows that $x \in M$.
- X is a *sequential space* if every sequentially closed subset of X is closed.
- X is *Fréchet–Urysohn* provided that for any point x in the closure of a subset M, there exists a sequence in M converging to x.
- X is *angelic* if its relatively countably compact subsets are relatively compact and the compact subspaces of X are Fréchet–Urysohn.
- X is *strictly angelic* if it is angelic and the separable compact subspaces of X are first countable
- X has *countable tightness* provided that for any point x in the closure of a subset M, there exists a countable set $M_x \subset M$ such that x is in the closure of M_x.
- X is a k-space if a subset $M \subset X$ is closed in X provided $M \cap K$ is closed in K for every compact subset K.
- X is *bisequential* if whenever a filter base $\mathcal{F}$ accumulates at $x \in X$, there exists a decreasing sequence of nonempty[3] subsets of X, $\{A_n;\ n \in \mathbb{N}\}$ which considered as a filter base converges to x, and is such that $A_n \cap F \neq \emptyset$, $\forall n \in \mathbb{N}$ and $\forall F \in \mathcal{F}$.

[3] From now on we omit the word nonempty in this context.

- X is *countably bisequential* [24, Definition 4.D.1] if whenever a countable filter base $\mathcal{F}$ accumulates at $x \in X$, there exists a decreasing sequence $\{A_n;\ n \in \mathbb{N}\}$ of nonempty subsets of X, which considered as a filter base converges to x, and is such that $A_n \cap F \neq \emptyset$, $\forall n \in \mathbb{N}$ and $\forall F \in \mathcal{F}$.

In [24] it is mentioned that Siwiec introduced the term *strongly Fréchet–Urysohn* to denominate the countably bisequential spaces because of the equivalence expressed in the next lemma. We will use both terms for this notion, depending on which properties we want to underline.

Lemma 1.1. [24, 4.D.2] *Let X be a topological space. The following statements are equivalent*:

(i) *X is countably bisequential*;
(ii) *If $\{A_n;\ n \in \mathbb{N}\}$ is a decreasing sequence accumulating at $x \in X$, then there exists $x_n \in A_n$ such that $x_n \to x$.*

The following picture illustrates the main relationships among the above notions:

Metrizable $\overset{(1)}{\Longrightarrow}$ First countable $\overset{(2)}{\Longrightarrow}$ Bisequential $\overset{(3)}{\Longrightarrow}$ Countably bisequential $\overset{(4)}{\Longrightarrow}$ Fréchet–Urysohn $\overset{(5)}{\Longrightarrow}$ Sequential $\overset{(6)}{\Longrightarrow}$ Countable tightness.

Metrizable $\overset{(7)}{\Longrightarrow}$ Strictly angelic $\overset{(8)}{\Longrightarrow}$ Angelic.

The implications (1), (3), (7) and (8) are trivial. Observe that if X is first countable the condition of bisequentiality is satisfied: for a filter base $\mathcal{F}$ that accumulates at $x \in X$, take as $\{A_n;\ n \in \mathbb{N}\}$ a decreasing sequence of members of a countable neighborhood basis of x. This proves (2).

In order to prove (4), assume that X is a countably bisequential space. Take a subset $A \subset X$ and any $x \in \overline{A}$. For the decreasing sequence formed by $A_n = A$ for all $n \in \mathbb{N}$, by (ii) in Lemma 1.1, a sequence $x_n \in A$ can be found such that $x_n \to x$. Thus, X is Fréchet–Urysohn.

In [15, Theorem 1.6.14] it can be seen a proof of implication (5) and in 1.7.13(c) of the same reference there are hints to prove (6). We give below a direct and simple proof of this fact.

Proposition 1.2. *Any sequential space has countable tightness.*

Proof. Assume X is sequential and take $M \subset X$. For any $x \in \overline{M}$ we must prove that there exists a countable subset, say $M_x \subset M$ such that $x \in \overline{M_x}$. Call $N := \bigcup\{\overline{A};\ A \subset M$, and $\operatorname{card} A \leqslant \aleph_0\}$. Clearly $N \subset \overline{M}$. We prove now that N is sequentially closed. Take $\{x_n,\ n \in \mathbb{N}\} \subset N$ such that $x_n \to y \in X$. For every $n \in \mathbb{N}$ there is a countable set $A_n \subset M$ such that $x_n \in \overline{A_n}$. Therefore $y \in \overline{\bigcup_{n\in\mathbb{N}} A_n}$, and by the definition of N, $y \in N$. Since X is sequential, N is closed, and so $x \in N = \overline{M}$. □

Next we state the double sequence property for a topological space X, introduced by Arhangelskii in [2]:

(α_4) For any family $\{x_{m,n};\ (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m = 1, 2, \ldots$, it is possible to choose a sequence $(i_k)_{k\in\mathbb{N}}$ of distinct natural numbers and a sequence $(j_k)_{k\in\mathbb{N}}$ of natural numbers such that $\lim_k x_{i_k, j_k} = x$.

It is proved in [25, Theorem 4] that a Fréchet–Urysohn topological group satisfies (α_4), although "topological group" cannot be replaced here by "topological space". Earlier in [5, Lemma 3.3, p. 99] (see also [31, p. 140]), it was observed that a Fréchet–Urysohn topological vector space has the following stronger property:

(AS) For any family $\{x_{m,n};\ (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$, with $\lim_n x_{m,n} = x \in X$, $m = 1, 2, \ldots$, it is possible to choose *strictly increasing* sequences of natural numbers $(i_k)_{k\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$, such that $\lim_k x_{i_k, j_k} = x$.

Lemma 1.3. *A Fréchet–Urysohn Hausdorff topological group G satisfies* (AS) *and hence* (α_4) *as well.*

Proof. Denote by 0 the neutral element of G; it is enough to show (AS) for any family $\{x_{m,n};\ (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset G$ with $\lim_n x_{m,n} = 0$, $m = 1, 2, \ldots$. Fix a sequence $(a_m)_{m\in\mathbb{N}} \subset G$ with $\lim_m a_m = 0$ and $a_m \neq 0$, $m = 1, 2, \ldots$ (if such a sequence does not exist, then G is a discrete space and the conclusion of the lemma holds trivially). Put $y_{m,l} = a_m + x_{m,l+m}$ if $a_m + x_{m,l+m} \neq 0$ and $y_{m,l} = a_m$ otherwise. Consider the set

$$M := \{y_{m,l};\ (m,l) \in \mathbb{N} \times \mathbb{N}\}.$$

Then $0 \notin M$. Let us see that 0 is in the closure of M. Fix a neighborhood U_0 of 0 and take another neighborhood U of 0 such that $U + U \subset U_0$; since $\lim_m a_m = 0$, there is m such that $a_m \in U$, and $\lim_n x_{m,n} = 0$ implies that we can choose l such that $x_{m,l+m} \in U$; then $y_{m,l} \in U + U \subset U_0$.

Since 0 is in the closure of M and G is Fréchet–Urysohn, there is a sequence $(m_k, l_k)_{k\in\mathbb{N}}$ such that $\lim_k y_{m_k,l_k} = 0$.

Case 1. The sequence $(l_k)_{k\in\mathbb{N}}$ is bounded.

Taking a subsequence if necessary, we can suppose that $l_k = r$, $k = 1, 2, \ldots$ for some natural number r. Since $0 = \lim_k y_{m_k,l_k} = \lim_k y_{m_k,r}$ and $y_{m_k,r} \neq 0$, $k = 1, 2, \ldots$, we conclude that $\lim_k m_k = \infty$. Taking once more a subsequence, we can suppose that $m_1 < m_2 < \cdots$.

Subcase 1.1. The set $N_1 = \{k \in \mathbb{N}:\ y_{m_k,r} = a_{m_k}\}$ is infinite.

Write $N_1 = \{p_1, p_2, \ldots\}$ with $p_1 < p_2 < \cdots$. Then $a_{m_{p_k}} + x_{m_{p_k},r+m_{p_k}} = 0$, $k = 1, 2, \ldots$. As $\lim_k a_{m_{p_k}} = 0$, we get: $\lim_k x_{m_{p_k},r+m_{p_k}} = 0$.

If we set now $i_k := m_{p_k}$, $k = 1, 2, \ldots$ and $j_k := r + m_{p_k}$, $k = 1, 2, \ldots$, we get the strictly increasing sequences $(i_k)_{k\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$, such that $\lim_k x_{i_k,j_k} = 0$.

Subcase 1.2. The set $N_1 = \{k \in \mathbb{N}:\ y_{m_k,r} = a_{m_k}\}$ is finite.

The set $N_2 = \{k \in \mathbb{N}:\ y_{m_k,r} \neq a_{m_k}\}$ is infinite and we may write: $N_2 = \{q_1, q_2, \ldots\}$ with $q_1 < q_2 < \cdots$. Then $y_{m_{q_k},l_{q_k}} = a_{m_{q_k}} + x_{m_{q_k},r+m_{q_k}}$, $k = 1, 2, \ldots$. As $\lim_k y_{m_{q_k},l_{q_k}} = 0$ and $\lim_k a_{m_{q_k}} = 0$, we get: $\lim_k x_{m_{q_k},r+m_{q_k}} = 0$.

If we set $i_k := m_{q_k}$, $k = 1, 2, \ldots$ and $j_k := r + m_{q_k}$, $k = 1, 2, \ldots$, we get again the strictly increasing sequences $(i_k)_{k\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$, such that $\lim_k x_{i_k,j_k} = 0$.

Case 2. The sequence $(l_k)_{k\in\mathbb{N}}$ is not bounded.

We may suppose that $(l_k)_{k\in\mathbb{N}}$ is strictly increasing. Then $\lim_k m_k = \infty$. Otherwise, taking a subsequence if necessary, we can state that $m_k = s$, $k = 1, 2, \ldots$ for some s. Since $(l_k)_{k\in\mathbb{N}}$ is strictly increasing, we have $\lim_k x_{s,s+l_k} = 0$. From $\lim_k y_{s,l_k} = 0$, we get $a_{m_k} = a_s = 0$, which contradicts our choice of $(a_m)_{m\in\mathbb{N}}$. So, $\lim_k m_k = \infty$. Then for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers we have $m_{n_1} < m_{n_2} < \cdots$. Set $i_k := n_k$, $k = 1, 2, \ldots$ and $j_k := m_{n_k} + l_{n_k}$, $k = 1, 2, \ldots$. Then $(i_k)_{k\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$ are strictly increasing sequences of natural numbers such that $\lim_k x_{i_k,j_k} = 0$. □

Remark 1. In [5, Problem 5, p. 111] it is posed the next question: let E be a topological vector space satisfying (AS), is then E a Fréchet–Urysohn space? The answer to a similar question is negative in the context of topological groups. In fact, in [25, Example 5] it is shown that a countable group may satisfy the following stronger property ($*$), without being even sequential.

($*$) For $x \in X$ and for a family $\{x_{m,n};\ (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x$, $m = 1, 2, \ldots$, it is possible to choose a sequence $(j_k)_{k\in\mathbb{N}}$ of natural numbers such that $\lim_k x_{k,j_k} = x$.

The first claim of the following proposition was stated in [2]:

Proposition 1.4. *A topological space X is strongly Fréchet–Urysohn if and only if it is Fréchet–Urysohn and has the double sequence property (α_4). Consequently, if X is a topological group, Fréchet–Urysohn and strongly Fréchet–Urysohn are equivalent properties.*

Proof. Both implications can be proved in a very natural way; we prove one of them, and leave the other for the reader. The last assertion follows from Lemma 1.3.

($\Leftarrow$) Let $\{F_n\}$ be a decreasing sequence such that $x \in \bigcap_{n\in\mathbb{N}} \overline{F}_n$. Since $x \in \overline{F}_m$, and X is Fréchet–Urysohn, a sequence can be taken $\{x_{m,n};\ n \in \mathbb{N}\} \subset F_m$ such that $x_{m,n} \to x$. By the property (α_4), there exist a sequence $\{i_k\}$ of distinct natural numbers and a sequence $\{j_k\}$ of natural numbers such that $x_{i_k,j_k} \to x$. Now the subsets $A_n := \{x_{i_k,j_k};\ k = n+1, n+2, \ldots\}$ constitute a decreasing sequence which accumulates at x and satisfies $A_n \cap F_m \neq \emptyset$, for all $n, m \in \mathbb{N}$. □

Lemma 1.5. *Let X be a topological space.*

(a) *If X is sequential, then it is a k-space.*
(b) *If X is a Hausdorff k-space and its compact subsets are sequential* (*in particular first countable or metrizable*), *then X is sequential.*

Proof. (a) is easy to verify.

(b) Take $L \subset X$ sequentially closed. Let us prove that $L \cap K$ is closed for every compact subset $K \subset X$. Since K is sequential, it is enough to see that whenever a sequence $\{x_n;\ n \in \mathbb{N}\} \subseteq L \cap K$ converges to some $x \in X$, then x must lie in $L \cap K$. But this follows from the facts that L is sequentially closed, and $K \subset X$ is closed. □

We deal now with the finitely multiplicative behaviour of sequential-like properties. A nice proof of the fact that the product of a metrizable space with a Fréchet–Urysohn may be even nonsequential is obtained in [14]. By the following theorem we can claim that a substantial improvement of this behaviour is obtained when requiring some algebraic structure in one of the factors.

Theorem 1.6. *Let X, Y be topological spaces.*

(a) [24, Proposition 4.D.4] *If X is bisequential and Y is countably bisequential, then $X \times Y$ is countably bisequential.*
(b) *If X is first countable and Y is a Fréchet–Urysohn space with the property (α_4), then $X \times Y$ is Fréchet–Urysohn.*
(c) *If X is a first countable space and Y is a Fréchet–Urysohn topological group, then $X \times Y$ is a Fréchet–Urysohn topological space.*

Proof. (b) As we have already noted, a first countable space is bisequential. Since Y is a Fréchet–Urysohn topological space with (α_4), from Proposition 1.4 it follows that Y is countably bisequential (or what is the same, strongly Frechet–Urysohn). Now from (a) we obtain that $X \times Y$ is a countably bisequential space, and by Lemma 1.1 it is Fréchet–Urysohn.

(c) follows from (b) because a Fréchet–Urysohn topological group satisfies (α_4) by Lemma 1.3. □

Remark 2. Earlier, in [5, Proposition 3.2, p. 99], the following weaker version of Theorem 1.6(c) was proved: if X is a metrizable topological vector space and Y is a Fréchet–Urysohn topological vector space, then $X \times Y$ is a Fréchet–Urysohn space. In this context observe that the product of two Frechét–Urysohn topological groups may not be Frechét–Urysohn. In [29] it is proved that the product of two Frechét–Urysohn locally convex spaces may fail to have countable tightness.

Our next statement gives a source of angelic sequential spaces with additional nice properties.

Theorem 1.7. *Let G be a metrizable Abelian topological group. The following statements hold*:

(a) *$G^\wedge_c$ is a hemicompact k-space. Moreover, $G^\wedge_c$ is a complete locally quasi-convex topological Abelian group.*
(b) *If G is furthermore separable, then $G^\wedge_c$ is a strictly angelic sequential space whose compact subsets are metrizable.*

Proof. (a) is known (see [8] and [4, Corollary 4.7, Proposition 4.11]).

(b) Let D be a countable dense subset in G and let $\mathcal{T}_D$ be the topology in $G^\wedge$ of pointwise convergence on D. Then $\mathcal{T}_D$ is metrizable and coarser than the compact open topology. Thus, in the compact subsets of $G^\wedge_c$ they coincide. By (a) and Lemma 1.5(b) we get that $G^\wedge_c$ is a sequential space. From [9, Proposition 9] it follows that $G^\wedge_c$ is strictly angelic. □

The assumption of separability cannot be dropped in Theorem 1.7(b). In fact, if G is an uncountable discrete group, then $G^\wedge_c$ is a compact group which may be neither sequential nor angelic, as the following example shows. Take G as the direct sum of c copies of $\mathbb{Z}$ with the discrete topology (in what follows we refer the reader to [7] for dualities between direct sums and products of topological Abelian groups). Its dual group is the product $\mathbb{T}^c$, which is compact but not sequentially compact [9, Example 28(1)], therefore non-angelic. On the other hand, the corresponding Σ-product is a sequentially closed, non-closed subgroup.

2. The main theorems

In this section we deal with further relationships among the sequential-like properties so far cited. We keep in mind especially the questions of for which classes of topological groups metrizability is equivalent to the Fréchet–Urysohn property and of when the latter differs from sequentiality.

It is known that in general a Fréchet–Urysohn Hausdorff locally convex space may not be metrizable. Seemingly the first example of this sort appeared in [6]. Many other examples of non-metrizable Fréchet–Urysohn locally convex spaces are contained in Proposition 2.1 below. For its formulation we recall some standard notation. For a topological space X, $C_p(X)$ (respectively $C_c(X)$) is the space of continuous real valued functions equipped with the topology of pointwise convergence (respectively with the compact-open topology).

Proposition 2.1. *Let X be a Tychonoff space. Then*:

(a) *$C_p(X)$ is Fréchet–Urysohn $\Leftrightarrow$ it is sequential $\Leftrightarrow$ it is a k-space* [1, Theorem II.3.7].
(b) *If X is compact, then $C_p(X)$ is Fréchet–Urysohn $\Leftrightarrow$ $C_p(X)$ is a k-space $\Leftrightarrow$ X is scattered* [1, Theorem III.1.2] (*the same equivalences hold if X is K-analytic* [11, Corollary 4.2]).
(c) *$C_c(X)$ is Fréchet–Urysohn $\Leftrightarrow$ it is sequential $\Leftrightarrow$ it is a k-space* [26, Theorem 5.1].
(d) *If X is first countable, then $C_c(X)$ is Fréchet–Urysohn $\Leftrightarrow$ X hemicompact* [16, Theorems 1, 2]).

It is known that every locally compact group with countable tightness (in particular, a compact Fréchet–Urysohn group) is metrizable [19], however a countably compact Fréchet–Urysohn topological group may not metrizable [26, Theorem 2.7]. The Malyhin's problem [26, Problem 3.11]: *is every countable Fréchet–Urysohn topological group metrizable*? remains open. In [30, Theorem 7.3] it is asserted that *a countable Fréchet–Urysohn topological group is metrizable iff its topology is analytic*. In [9] under (CH) it is shown that every compact sequentially compact topological group is metrizable; the same claim can be stated as an axiom weaker than (CH) [13].

In [10,20] a wide class of topological vector spaces for which Fréchet–Urysohn implies metrizability is considered. Next, we see that the dual of a metrizable group also has this property.

Theorem 2.2. *For a metrizable topological group G the following statements are equivalent*:

(i) *$G_c^\wedge$ is Fréchet–Urysohn.*
(ii) *$G_c^\wedge$ is a locally compact* metrizable *space.*

Proof. (i) $\Rightarrow$ (ii). By Theorem 1.6(c), $G \times G_c^\wedge$ is Fréchet–Urysohn, and in particular a k-space. Therefore, the evaluation mapping $w : G \times G_c^\wedge \to \mathbb{T}$ ($w(x,\varphi) = \varphi(x)$), which is always k-continuous, is continuous. According to [23, Proposition 1.2] (see also [22]), we have that $G_c^\wedge$ is locally compact.

Next we show that $G_c^\wedge$ is metrizable.

Step 1. If K is a compact subgroup of $G_c^\wedge$, then K is metrizable.

In fact, every compact group is dyadic (in Abelian case, which is sufficient for our aims, this result can be found in [17, (25.35)]). Since every dyadic Hausdorff space with countable tightness is second countable (see [3, Theorem 7, p. 1223] or [15, 3.12.12(h), p. 231]), we get that K is metrizable.

Step 2. The metrizability of $G_c^\wedge$ will be obtained by an argument similar to that of [9, Proposition 12]. Since $G_c^\wedge$ is locally compact, by [12, Theorem 3.3.10] there exists a closed subgroup H of $G_c^\wedge$ containing an open compact subgroup, say K, such that $G_c^\wedge$ is topologically isomorphic to the product $\mathbb{R}^n \times H$ for some $n \in \mathbb{N} \cup \{0\}$. By step 1, K is metrizable. Since K is open in H, it follows that H is metrizable and therefore $G_c^\wedge$ is metrizable. □

Corollary 2.3. *Let E be a metrizable dually separated topological vector space. The following statements are equivalent*:

(i) E_c^* *is Fréchet–Urysohn.*
(ii) E *is finite-dimensional.*

Proof. (i) ⇒ (ii). In [28] it is proved that E_c^* is topologically isomorphic to $E_c^\wedge$. Thus, by the implication (i) ⇒ (ii) of Theorem 2.2, the Hausdorff topological vector space E_c^* is locally compact and consequently finite-dimensional. Since E is dually separated, E is also finite dimensional. □

Theorem 2.4. *Let G be a metrizable separable topological Abelian group for which $G_c^\wedge$* is not *locally compact. Then $G_c^\wedge$ is a complete strictly angelic hemicompact sequential* non-Fréchet–Urysohn *locally quasi-convex group.*

Proof. From Theorem 1.7 we obtain the positive claims about $G_c^\wedge$. Since $G_c^\wedge$ is not locally compact, it is not Fréchet–Urysohn by (i) ⇒ (ii) of Theorem 2.2. □

Corollary 2.5. *The direct sum of countably many copies of $\mathbb{T}$ endowed with the box topology, say $\omega\mathbb{T}$, is a sequential hemicompact complete topological group, which is not Fréchet–Urysohn.*

Proof. Clearly $\omega\mathbb{T}$ is topologically isomorphic with $G_c^\wedge$, where $G := \mathbb{Z}^{\mathbb{N}}$ [7]. Since G is metrizable and separable and $\omega\mathbb{T}$ is not locally compact (otherwise $(\omega\mathbb{T})_c^\wedge$, which is topologically isomorphic with $\mathbb{Z}^{\mathbb{N}}$, would be locally compact), Theorem 2.4 applies. □

The following consequence of Theorem 2.4 looks more impressive.

Theorem 2.6. *Let E be a metrizable separable topological vector space for which E^* is infinite-dimensional. Then E_c^* is a sequential hemicompact strictly angelic complete non-Fréchet–Urysohn locally convex space whose compact subsets are metrizable.*

Corollary 2.7. *The locally convex direct sum of countably many copies of $\mathbb{R}$, say $\omega\mathbb{R}$, is a sequential hemicompact complete countable dimensional non-Fréchet–Urysohn locally convex space whose compact subsets are metrizable.*

Proof. Clearly $\omega\mathbb{R}$ is topologically isomorphic with E_c^*, where $E := \mathbb{R}^{\mathbb{N}}$ [18, Theorem 8.8.5]. Since E is metrizable and separable and $\omega\mathbb{R}$ is not finite-dimensional, Theorem 2.6 applies. □

An analogue of the previous theorem can be stated for the strong dual, as we do next.

Theorem 2.8. *For a metrizable locally convex space E the following statements are equivalent*:

(i) E *is normable*;
(ii) E_β^* *is Fréchet–Urysohn.*

Proof. (i) ⇒ (ii) is clear, since under this assumption E_β^* is also normable.

In order to prove (ii) ⇒ (i), observe that $E \times E_\beta^*$ is Fréchet–Urysohn by Theorem 1.6(c), and the evaluation $e : E \times E_\beta^* \to \mathbb{K}$ is sequentially continuous. Therefore e is continuous, and this implies that E is normable (this is a well-known fact, see e.g. [18, Theorem 9.1.3(a)]). □

Remark 3. (a) The result of Corollary 2.7 was already known (see [25, Example 1], where a direct proof is given).

(b) Theorem 2.8 can be derived also from the results of [10]. Our proof is different.

(c) In connection with Theorem 2.8 we note that for a metrizable locally convex space E the space E_β^* is *sequential* iff either *it is normable* or *it is Montel* [21, Theorem 4.5].

Acknowledgements

The authors are grateful to the referee for his thorough reading of the paper and for the improvements suggested. Also to Professor Van Mill for his kind encouragement along the process of publication.

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