ON THE RELATION BETWEEN FUZZY PREORDERS, FUZZY CLOSING MORPHOLOGICAL OPERATORS AND FUZZY CONSEQUENCE OPERATORS

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Abstract

In a previous paper [10] we explored the notion of coherent fuzzy consequence operator. It is well-known that the operator induced by a fuzzy preorder through Zadeh's compositional rule is always a coherent fuzzy consequence operator. It is also known that the relation induced by a fuzzy consequence operator is a fuzzy preorder if such operator is coherent [7]. Fuzzy closing operators of mathematical morphology can be considered as fuzzy consequence operators. In [12] we showed that they are coherent operators. The aim of this paper is to analyze the relations between both classes of operators and the class of all fuzzy preorders in order to translate well know properties from Approximate Reasoning to the one of Image Processing.

Keywords: Fuzzy Preorder, Fuzzy Consequence Operator, Fuzzy Morphological Closing Operator, Coherent Operators, Approximate Reasoning, Image Processing.

1 INTRODUCTION

Consequence Operators in classical logic were introduced by A. Tarski in 1930 [20]. According to Tarski, a logic is just a set of propositions with a consequence operator. The relationship between consequence operators and preorders is well known [5].

Concepts of Fuzzy Preorder and Fuzzy Consequence Operator (FCO for short) are essential on fuzzy logic. These notions have been defined as a natural generalization of the classical ones:

Given a non-empty universal set X which will represent a set of propositions and a t-norm $*$, a fuzzy (binary) relation R on X (fuzzy subset of $X \times X$) is called a fuzzy $*$-preorder if it satisfies:

(R1) $R(x,x) = 1 \ \forall \ x \in X$ (reflexivity)

(R2) $R(x,z) \geq R(x,y) * R(y,z) \ \forall \ x,y,z \in X$ ($*$-transitivity).

If R is also symmetric, this is, $R(x,y) = R(y,x)$ for all $x,y \in X$ then it is called a fuzzy $*$-similarity, fuzzy $*$-indistinguishability or, equivalently, fuzzy $*$-equivalence. If R is only reflexive and symmetric, we will say that it is a fuzzy tolerance.

Fix a complete lattice L which will be the range of the memberships of the fuzzy subsets of X, J. Pavelka introduced in 1979 the concept of FCO on X in fuzzy logic [18] extending the concept of consequence operator in Tarski's sense in a natural way.

A function $C : L^X \longrightarrow L^X$ is a FCO on X if it satisfies:

(C1) $\mu \subset C(\mu)$ for all fuzzy subset $\mu \in L^X$ (inclusion)

(C2) $\mu_1 \subset \mu_2 \Longrightarrow C(\mu_1) \subset C(\mu_2)$ for all $\mu_1, \mu_2 \in L^X$ (monotony)

(C3) $C(C(\mu)) \subset C(\mu)$ for all $\mu \in L^X$ (idempotence)

Notice that, under the inclusion axiom, (C3) may be written equivalently as (C3') $C(C(\mu)) = C(\mu) \ \forall \ \mu \in L^X$.

These operators, also called closure operators, have been studied extensively [1],[2],[8],[12],[13],[24]. During the last and current decades, these operators have been also studied in the context of the fuzzy logic taking the chain $L = [0,1]$ as a special case [6],[7],[10],[11],[15].

In 1991, J.L. Castro and E. Trillas proved [7] the following result: If R is a fuzzy $*$-preorder then the operator C_R^* between fuzzy subsets of X given by the max-$*$ Zadeh's compositional rule

$$C_R^*(\mu) = \mu \circ^* R \quad (1)$$

is a fuzzy consequence operator (induced by R), where

$$(\mu \circ^* R)(x) = \sup_{w \in X} \{\mu(w) * R(w, x)\} \quad (2)$$

We proved [10] that the converse of the previous result is also true: R is a $*$-preorder if and only if C_R^* is a FCO. Thus the only possibility to obtain FCOs from relations using Zadeh's max-$*$ compositional rule is to work with $*$-preorders.

For the inverse process (inducing fuzzy preorders from FCOs), J.L. Castro and E. Trillas added the coherence axiom to the FCO concept [6]. A FCO C is called $*$-coherent if

$$\mu(a) * C(\varphi_a)(x) \leq C(\mu)(x) \quad (3)$$

for all $\mu \in [0,1]^X$ and for all $(a, x) \in X \times X$, where $\varphi_a(t) = \varphi_{\{a\}}(t)$ is the crisp membership of the singleton $\{a\}$.

It is proven [7] that if C is a $*$-coherent FCO then the relation defined by

$$R_C(x, y) = C(\varphi_x)(y) \quad (4)$$

is a fuzzy $*$-preorder (induced by C). It is also proven that the operator C_R^* is $*$-coherent.

It is easy to extend the notion of coherence to operators $C : L^X \longrightarrow L^X$ between fuzzy subsets in L^X. This is: a FCO C is called $*$-coherent if $\mu(a) * C(\varphi_a)(x) \leq C(\mu)(x)$ for all $\mu \in L^X$ and for all $(a, x) \in X \times X$, where $\varphi_a(t) = \begin{cases} 1 & if\ t = a \\ 0 & if\ t \neq a \end{cases}$, being in this case 1 and 0 the greatest and least element of the lattice L, respectively. Then, all the previous results hold.

On the other hand, the classical morphological operators of image processing have been introduced by G. Matheron an J. Serra in [16],[17],[21]. These operators have been studied recently, see for instance [12],[14],[23]. In [14] they are studied in a new context: obtaining relevant information in fuzzy relational systems.

In this context, the most used morphological operators are erosion, dilation, opening and closing that are given respectively by $\varepsilon_R(\mu) = R^{op} \lhd \mu$, $\delta_R(\mu) = R \circ \mu$, $\alpha_R(\mu) = R \circ (R^{op} \lhd \mu)$ and $\beta_R(\mu) = R \lhd (R^{op} \circ \mu)$, where R is the relation acting as structurant element and μ is in the class of all fuzzy subsets of the universal set X or $X \times X$. R^{op} denotes the inverse relation of R, $R^{op}(x, y) = R(y, x)$ and the operators $\circ$, $\lhd$ are defined as usual:

$(R \circ \mu)(x) = \sup_{w \in X} \{R(x, w) * \mu(w)\}$, $(R \lhd \mu)(x) = \inf_{w \in X} \{R(x, w) \rightsquigarrow \mu(w)\}$ for some t-norm $*$ and for some implication operator $\rightsquigarrow$.

Notice that the operator $\circ$ and possibly the operators $\lhd$ and $\rightsquigarrow$ depend on the t-norm $*$. However, when there is no confusion, in order to alleviate the notation we omit the symbol $*$.

From now on, $*$ will be a left-continuous t-norm, that is, $\alpha * (\sup M) = \sup(\alpha * M)$ for all $\alpha \in L$, $M \subset L$ and $\rightsquigarrow_*$ will denote the residuated implication induced by the t-norm $*$: $\alpha \rightsquigarrow_* \beta = \sup\{w \in L / \alpha * w \leq \beta\}$. It is also called the pseudoinverse application of $*$.

Both in the crisp and in the fuzzy cases, opening and closing operators with an isotropic structuring element are used in image processing to eliminate specific image details smaller than the structuring element. The global shape of the objects is not distorted.

In particular, a closing operator connects objects that are close to each other, fills up small holes and smooths the object outline by filling up narrow gulfs. Meanings of *near*, *small* and *narrow* are related to the size and the shape of the structuring element.

Figure 1 ([22]) shows an example where a set is transformed by a closing operator through a disk as structuring element.

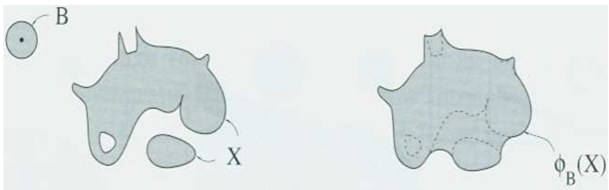

Figure 1: Effect of a closing operator on a non-smooth object

In this way, the concept of fuzzy closing operator φ_R^* induced by a fuzzy relation is relevant in fuzzy mathematical morphology, defined in [12] as follows. For any fuzzy relation R on X,

$$\varphi_R^*(\mu) = R^{op} \lhd_{\rightsquigarrow_*} (R \circ^* \mu) \quad (5)$$

where $R \circ^* \mu$ is given by

$(R \circ^* \mu)(x) = \sup_{w \in X} \{R(x, w) * \mu(w)\}$ and

$(R \lhd_{\rightsquigarrow_*} \mu)(x) = \inf_{w \in X} \{R(x,w) \rightsquigarrow_* \mu(w)\}$.

It is well-know that φ_R^* is a closure operator, that is, a fuzzy consequence operator.

In Section 2, we recall some properties which are proven in [12].

In Section 3, we show some results about the relationship between fuzzy preorders and their induced consequence and closing morphological fuzzy operators.

2 FUZZY CLOSING MORPHOLOGICAL OPERATORS AS FUZZY CONSEQUENCE OPERATORS

In [12] we proved that, for all t-norm $*$, the fuzzy closing operator induced by a relation is always a $*$-coherent operator acting as a fuzzy consequence operator. More precisely, we proved the following result.

Theorem 1.- ([12]) *Let R be any fuzzy relation on X. Then the fuzzy closing operator $\varphi_R^* : L^X \longrightarrow L^X$ induced by R by means of (5) is a $*$-coherent fuzzy consequence operator.*

Since the relation induced by a coherent fuzzy consequence operator is a preorder, it immediately follows that the relation induced by the fuzzy closing operator φ_R^* is a preorder:

Corollary 1.- ([12]) *Let R be a fuzzy relation on X. Then the fuzzy relation induced by the operator φ_R^* by means of (4) is a $*$-preorder.*

We also proved the following characterization of such induced relation from which other properties can be easily deduced.

Lemma 1.- ([12]) *Let R be any fuzzy relation on X and put $C = \varphi_R^*$. Then the $*$-preorder induced by C by means of (4), R_C is such that*

$$R_C = (R^{op} \lhd_{\rightsquigarrow_*} R)^{op} \tag{6}$$

Recall that the $\lhd_{\rightsquigarrow_*}$ composition for two relations R and S on X is defined by $(R \lhd_{\rightsquigarrow_*} S)(x,y) = \inf_{w \in X} \{R(x,w) \rightsquigarrow_* S(w,y)\}$.

From the previous lemma it follows immediately that $(R^{op} \lhd_{\rightsquigarrow_*} R)^{op}$ is always a $*$-transitive relation, then $R^{op} \lhd_{\rightsquigarrow_*} R$ is also $*$-transtitive. This result about the transitivity of the relation R does not involve the composition operator $\circ$, even though transitivity is defined using composition.

The following characterization about the transitivity of a relation R does not involve composition either.

Theorem 2.- ([12]) *Let R be any fuzzy relation on X. Then R is $*$-transitive if and only if $R \leq R^{op} \lhd_{\rightsquigarrow_*} R$.*

Accordingly, we have the following characterization of fuzzy preorders without the operator $\circ$:

Corollary 2.- ([12]) *Let R be any fuzzy relation on X. Then R is an $*$-preorder if and only if $R = R^{op} \lhd_{\rightsquigarrow_*} R$, this is, R $*$-preorder if and only if $R_C = R^{op}$.*

3 FUZZY PREORDERS AND THEIR INDUCED CONSEQUENCE AND CLOSING MORPHOLOGICAL FUZZY OPERATORS

From now on, we make use of the following notation introduced in [11].

Γ^* will represent the class of all fuzzy preorders, Ω will represents the class of all fuzzy consequence operators, $\tilde{\Omega}^*$ the subclass of all $*$-coherent fuzzy consequence operators and Ω_p^* the subclass of all fuzzy consequence operators induced by a $*$-preorder by means of (1), this is, $\Omega_p^* = \big\{C \in \Omega \mid \exists\, R \in \Gamma^*,\, C = C_R^*\big\}$.

$\theta^* : \Gamma^* \to \Omega$ is the function such that for each $*$-preorder R, $\theta^*(R) = C_R^*$ given by means of (1) and $\tilde{\theta} : \tilde{\Omega}^* \to \Gamma^*$ is the function such that for each $*$-coherent fuzzy consequence operator C, $\tilde{\theta}(C) = R_C$ given by means of (4). Notice that function $\tilde{\theta}$ does not depend of the t-norm $*$.

Here are some elementary properties of these families and functions.

It is easy to prove that θ^* is one to one. Moreover for each fuzzy relation R, $R_{C_R^*}$ is exactly the relation R, in other words, $\tilde{\theta} \circ \theta^*$ is the identity mapping on Γ^*.

In consequence, if $C \in \Omega_p^*$, there exists a unique $R \in \Gamma^*$ such that $\theta^*(R) = C$, then $\tilde{\theta} \circ \theta^*(R) = R$ and $\theta^* \circ \tilde{\theta}(C) = (\theta^* \circ \tilde{\theta}) \circ \theta^*(R) = \theta^* \circ (\tilde{\theta} \circ \theta^*)(R) = \theta^*(R) = C$. Conversely, if $\theta^* \circ \tilde{\theta}(C) = C$, C is the image of the fuzzy relation $\tilde{\theta}(C)$ by the mapping θ^* and $C \in \Omega_p^*$. Therefore, $\Omega_p^* = \big\{C \in \Omega \mid \theta^* \circ \tilde{\theta}(C) = C\big\}$.

Now we will prove the following lemma in order to obtain a characterization of $*$-preorders which involves fuzzy closing morphological operators that are induced by a $*$-preorder as fuzzy consequence operators in Approximate Reasoning by means of (1).

Lemma 2.- *Let C be any $*$-coherent operator (satisfying (3), not necessarily fuzzy consequence operator). Then $\theta^* \circ \tilde{\theta}(C) \subset C$.*

Proof.- If C satisfies (3):

$$\theta^* \circ \tilde{\theta}(C)(\mu)(x) = C^*_{\tilde{\theta}(C)}(\mu)(x) =$$

$$\sup_{w \in X} \{\mu(w) * \tilde{\theta}(C)(w,x)\} =$$

$$\sup_{w \in X} \{\mu(w) * R_C(w,x)\} =$$

$$\sup_{w \in X} \{\mu(w) * C(\varphi_w)(x)\} \leq$$

$$\sup_{w \in X} \{C(\mu)(x)\} = C(\mu)(x)$$

Therefore $\theta^* \circ \tilde{\theta}(C) \subset C$. □

Theorem 3.- *Let R be any fuzzy relation. If R is a $*$-preorder then $\varphi^*_R \in \Omega^*_p$, this is, φ^*_R is induced by a $*$-preorder by means of (1).*

Proof.- Assume that R is $*$-preorder and prove that $\theta^* \circ \tilde{\theta}(\varphi^*_R) = \varphi^*_R$. From Theorem 1, φ^*_R is a $*$-coherent operator, then $\theta^* \circ \tilde{\theta}(\varphi^*_R) \subset \varphi^*_R$ by Lemma 2.

On the other hand,

$$\varphi^*_R(\mu)(x) = \inf_{w \in X} \{R(w,x) \rightsquigarrow_* C^*_{R^{op}}(\mu)(w)\} \leq$$

$$R(x,x) \rightsquigarrow_* C^*_{R^{op}}(\mu)(x) = 1 \rightsquigarrow_* C^*_{R^{op}}(\mu)(x) =$$
$$C^*_{R^{op}}(\mu)(x)$$

From Corollary 2, if R $*$-preorder then

$$R_C = \tilde{\theta}(\varphi^*_R) = R^{op}$$

Thus

$$\varphi^*_R \subset C^*_{R^{op}} = \theta^*(R^{op}) =$$
$$\theta^*(\tilde{\theta}(\varphi^*_R)) = \theta^* \circ \tilde{\theta}(\varphi^*_R) \quad \square$$

Notice that if R is a $*$-preorder then $\varphi^*_R \in \Omega^*_p$ and there exists a $*$-preorder S such that $\varphi^*_R = C^*_S$. Thus

$$\tilde{\theta}(\varphi^*_R) = \tilde{\theta}(C^*_S) = \tilde{\theta} \circ \theta^*(S) = S$$

and

$$\theta^* \circ \tilde{\theta}(\varphi^*_R) = \theta^*(S)$$

As θ^* is one to one

$$\tilde{\theta}(\varphi^*_R) = S$$

that is

$$R_C = S$$

From Corollary 2, $R_C = R^{op}$ and $S = R^{op}$.

From this, the following corollary is straightforward.

Corollary 3.- *Let R be any fuzzy relation. If R is a $*$-preorder then $\varphi^*_R = C^*_{R^{op}}$.*

Notice that, for an equivalence relation R, it is enough to consider in Corollary 3, $R^{op} = R$ by symmetry to obtain the following result.

Corollary 4.- *Let R be any fuzzy relation. If R is a $*$-equivalence then $\varphi^*_R = C^*_R$.*

Finally, we show two properties for relations which are not transitive.

If R is only reflexive:

Theorem 4.- *Let R be any fuzzy relation. If R is reflexive then $\varphi^*_R \leq C^*_{R^{op}}$.*

Proof.- If R is reflexive then

$$\varphi^*_R(\mu)(x) = \inf_{t \in X} \{R(t,x) \rightsquigarrow_* \sup_{w \in X} \{R(t,w) * \mu(w)\}\} \leq$$

$$R(x,x) \rightsquigarrow_* \sup_{w \in X} \{R(x,w) * \mu(w)\}$$

By reflexivity

$$\varphi^*_R(\mu)(x) \leq 1 \rightsquigarrow_* \sup_{w \in X} \{R(x,w) * \mu(w)\} =$$

$$\sup_{w \in X} \{\mu(w) * R(x,w)\} = C^*_{R^{op}}(\mu)(x) \quad \square$$

If R is a tolerance:

Corollary 5.- *Let R be any fuzzy relation. If R is reflexive and symmetric then $\varphi^*_R \leq C^*_R$.*

4 CONCLUSIONS AND FUTURE WORKS

In [12], we have connected the class of fuzzy consequence operators in the context of approximate reasoning with the class of fuzzy closing operators used in image processing acting as filters.

We do not know of any previous relationship between both classes in the literature apart that both are closure operators. Nevertheless, an interesting relationship between several fuzzy mathematical morphologies, approximate reasoning and spatial reasoning is established [3],[4].

In this paper, we have proved some properties that connects both structures with the class of all fuzzy preorders.

More precisely, in Theorem 3 we have shown that if R is a preorder then the fuzzy closing morphological operator induced by R by means of (5) is also induced by R as fuzzy consequence operator by means of (1). In fact, we have proved that if R is $*$ preorder then $\varphi^*_R = C^*_{R^{op}}$.

From this, we have shown in Corollary 4 that for equivalence relations, the induced fuzzy closing morphological operator coincides with induced fuzzy consequence operator, this is, $\varphi^*_R = C^*_R$.

In future works we will use the connection established between both classes of operators and the class of preorders to translate well known properties from the field of approximate reasoning to the one of fuzzy relational systems. In particular we will explore its applications to image processing.

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