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Amplitude Equation for stationary convection in a rotating viscoelastic magnetic fluid

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We report both theoretical and numerical results on convection for a magnetic fluid under rotation in a viscoelastic carrier liquid. The viscoelastic properties are given by the Oldroyd model. We focus on the stationary convection for idealized boundary conditions. Close to the first bifurcation, the coefficients of the corresponding amplitude equation are determined. The effect of the Kelvin force and of the rotation on instability thresholds for a diluted suspension are also emphasized.

Introduction. Magnetic fluids are formed by a stable colloidal suspension of magnetic particles dispersed in a carrier liquid. The first studies on the convective instability for a rotating layer in a magnetic fluid have been reported by Gupta and Gupta [1] and by Venkatasubramanian and Kaloni [2]. An amplitude equation for the stationary convection with idealized boundary condition was derived in Ref. [3]. The Küppers-Lortz instability for the case of a rotating magnetic fluid was formulated by Auernhammer and Brand [4]. The thermal convection for rotating binary ferrofluids with idealized boundary condition has been investigated in Ref. [5]. In the stationary case an analytical expression was found for the Rayleigh number as function of the control parameters. In addition, the weakly nonlinear analysis for stationary convection in a rotating magnetic binary mixture was studied in Ref. [6].

Viscoelastic properties of fluids can be described by a constitutive equation, which relates the stress and strain rate tensors. Finding this relation, which should generalize the linear dependence characteristic of Newtonian fluids, is the main purpose of the science of rheology. The simplest constitutive equation capable of describing realistically the viscoelastic properties is given by the so-called Oldroyd model [7]. Convection in viscoelastic fluids has been studied by Parmentier et al. [8]. Recently, studies on stationary and oscillatory convection in viscoelastic magnetic fluid have been done [9–11]. Other important features of magnetic fluids, in both experimental and theoretical situations are discussed at length in Refs. [12, 13]

The aim of this paper is to present, as a preliminary result, the influence of the rotation in convective cells in viscoelastic magnetic fluids considering the case where the separation ratio and magnetic separation ratio are not too large such that the simple fluid approximation can be used and we do not need to include a description in terms of a binary liquid [14]. To this aim an Oldroyd viscoelastic magnetic fluids heated from below is considered. The description of the system involves many parameters whose values have not yet all been determined accurately. Therefore, we are left with some freedom in fixing the parameter values. Close to the bifurcation the weakly nonlinear analysis can be performed and an amplitude equation can be derived. For idealized boundary condition, we determine analytically the coefficients of the corresponding amplitude equation in the stationary bifurcation case. Finally, the nature of the bifurcation is discussed.

Model Equations. We consider a layer (thickness d) of incompressible magnetic fluid in a viscoelastic carrier liquid, with very large horizontal extension (xy-plane) in a vertical gravitational field $\mathbf{g}$ and subject to a vertical temperature gradient. The layer is rotating uniformly around the vertical axis, $\hat{\mathbf{z}}$, with uniform angular velocity ϖ. The magnetic fluid properties can be modeled as electrically nonconducting superparamagnets. The magnetic field $\mathbf{H}$ is assumed to be oriented also in the vertical direction. The magnetic field would be homogeneous, if the magnetic fluid were absent. Let us choose the z-axis such that $\mathbf{g} = -g\hat{\mathbf{z}}$ and let us assume that the layer has its interfaces at coordinates $z = -d/2$ and $z = d/2$. A static temperature difference across the layer is imposed, $T(z = -d/2) = T_0 + \triangle T$ and $T(z = d/2) = T_0$. Under the Boussinesq approximation, the equations for the dimensionless perturbations of the conductive state and the Maxwell equations read as

$$\nabla \cdot \mathbf{v} = \mathbf{0} \ , \tag{1}$$

$$P^{-1} d_t \mathbf{v} = -\nabla p + \nabla \cdot \overline{\boldsymbol{\tau}} + T_a^{1/2} \mathbf{v} \times \ \hat{\mathbf{z}} + Ra \boldsymbol{\Sigma} \ , \tag{2}$$

$$(1 + \Gamma D_t)\overline{\boldsymbol{\tau}} = (1 + \Gamma \Lambda D_t)\overline{\boldsymbol{D}} \ , \tag{3}$$

$$d_t(\theta - M_4 \partial_z \phi) = (1 - M_4)\upsilon_z + \nabla^2 \theta \ , \tag{4}$$

$$(\partial_{zz} + M_3 \nabla_\perp^2)\phi - \partial_z \theta = 0 \ , \tag{5}$$

$$\nabla^2 \phi_{ext} = 0 \ , \tag{6}$$

where $\mathbf{v} = \{\upsilon_x, \upsilon_y, \upsilon_z\}$ is the velocity field, p is the effective pressure which contains also the centrifugal contribution, $\overline{\boldsymbol{\tau}}$ is the stress tensor, θ is the temperature perturbation and ϕ denotes the magnetic potential. Here $\boldsymbol{\Sigma} = [(1+M_1)\theta - (M_1 - M_5)\partial_z\phi]\hat{\mathbf{z}} + M_1\theta\nabla(\partial_z\phi)$ and $\nabla_\perp^2 = \partial_{xx} + \partial_{yy}$ denotes the horizontal Laplacian operator. The time derivatives, $d_t f = \partial_t f + \mathbf{v} \cdot \nabla f$ indicates the total derivative and D_t denotes an invariant ("frame-indifferent") time derivative, defined as $D_t\overline{\boldsymbol{\tau}} = d_t\overline{\boldsymbol{\tau}} + \overline{\boldsymbol{\tau}} \cdot \overline{\mathbf{W}} - \overline{\mathbf{W}} \cdot \overline{\boldsymbol{\tau}} + a(\overline{\boldsymbol{\tau}} \cdot \overline{\mathbf{D}} + \overline{\boldsymbol{D}} \cdot \overline{\boldsymbol{\tau}})$, where $\overline{\boldsymbol{W}}$ and $\overline{\boldsymbol{D}}$ are the skew-symmetric part and the symmetric part of the velocity field gradient, respectively; a is a phenomenological parameter that lies in the range -1 to $+1$. For $a = -1$, one gets the lower convected Jeffrey's model (Oldroyd B), for $a = 0$ one gets the so-called corotational Jeffrey's model, and $a = 1$ describes the upper convected Jeffrey's model (Oldroyd A). Let us comment that the coefficient a is not completely independent of the other rheological parameters [15].

Importantly, in Eqs. $(1) - (6)$, the following groups of dimensionless numbers have been introduced: ***(a)*** (pure fluids) The Rayleigh number, $Ra = \alpha_T g \triangle T d^3/\kappa\nu$, accounting for buoyancy effects and the Prandtl number, $P = \nu/\kappa$, relating viscous and thermal diffusion time scales. ***(b)*** (rotation in pure fluids) The Taylor number $T_a = (2\varpi d^2/\nu)^2$. ***(c)*** (magnetic fluid) The strength of the magnetic force relative to buoyancy is measured by the parameter $M_1 = \beta\chi_T^2 H_0^2/(\rho_0 g \alpha_T(1+\chi))$; the nonlinearity of the magnetization, $M_3 = 1 - (\chi_H H_0^2)/(1+\chi)$, a measure of the deviation of the magnetization curve from the linear behavior $M_0 = \chi H_0$;

the relative strength of the temperature dependence of the magnetic susceptibility $M_4 = \chi_T^2 H_0^2 T_0 / c_H(1+\chi)$; and the ratio of magnetic variation of density with respect to thermal buoyancy $M_5 = \alpha_H \chi_T H_0^2/(\alpha_T(1+\chi))$. ***(d)***(viscoelastic fluid) The Deborah number, $\Gamma = \lambda_1 \bar{\kappa}/d^2$, and the ratio between retardation and stress relaxation times, $\Lambda = \lambda_2/\lambda_1$. Since $\lambda_{1,2}$ are both positive, it means that Γ and Λ are consequently also both positive. For $\Gamma = 0$ one recovers the Newtonian fluid while for $\Lambda = 0$ one describes a Maxwellian fluid. Let us comment on the numerical values of the parameters; the parameters Ra and T_a can be changed by several orders of magnitude, while a typical value for P in viscoelastic fluid is $P \sim 10^0 - 10^3$. The magnetic numbers have the following order of magnitude: $M_1 \sim 10^{-4} - 10$, $M_3 \gtrsim 1$, $M_4 \sim M_5 \sim 10^{-6}$ for typical magnetic field strengths [5,14]. For aqueous suspensions it is commonly accepted that the Deborah number is about $\Gamma \sim 10^{-3} - 10^{-1}$ [16], but for other kinds of viscoelastic fluids the Deborah number can be as high as $\Gamma \sim 10^3$. Unfortunately, no experimental data are available for either the retardation or the stress relaxation times, so we treat Λ as an arbitrary parameter in the range $[0,1]$. In addition, the above set of equations is still unnecessarily complicated. We will simplify it first by neglecting M_4, which is a common simplification in the description of instabilities in ferrofluids [14]. Since M_4 is not related to viscoelastic effects, which we are interested in here, we expect not to loose any reasonable aspect of the problem under consideration. The same is true for the coefficient M_5. So, the values of $\{M_4, M_5\}$ in the following analysis are taken as zero. Thus, we are left with two magnetic field dependent effects characterized by the parameters $\{M_1, M_3\}$. The first one denotes the influence of the Kelvin force and is expected to have the dominant influence on the convection behavior. The second parameter, M_3, is different from 1 due to the intrinsic nonlinearity of the magnetization and is only a weak function of the external magnetic field.

For the sake of simplicity, the analysis is limited to two-dimensional motion [17]. Therefore, we can use a description of the velocity field (v_x, v_z) in terms of a stream function $\psi(x,z,t)$. Due to its symmetry properties, the extra stress tensor has only three independent coefficients, namely: $\tau_{xx}, \tau_{zz}, \tau_{zx} = \tau_{xz}$. Instead of these individual components, the following three scalar quantities are usually considered in rheology: the trace $U = \tau_{xx} + \tau_{zz}$, the normal stress difference $S = \tau_{xx} - \tau_{zz}$ and the in-plane shear stress $\tau = \tau_{zz}$. Using the aforementioned assumptions, Eqs. (1) – (6) can be written in a compact form as

$$\boldsymbol{\mathcal{L}}\mathbf{u} + \boldsymbol{\mathcal{N}}\left(\mathbf{u} \mid \mathbf{u}\right) = 0 \tag{7}$$

being $\mathbf{u} = (\psi, \tau, U, S, \theta, \phi)^T$, and $\{\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{N}}\}$ stands for the linear and the nonlinear operators of the corresponding equations, respectively. We impose the following idealized boundary conditions $\psi = D^2\psi = \theta = D\phi = \tau = DS = U = 0$ at $z = \pm 1/2$; with $D^n f = \partial_z^n f$. In the next section, we present a weakly nonlinear analysis of the system (7) in the case of a stationary bifurcation.

Weakly Nonlinear Analysis. Since the linear instability threshold of the conducting state in the stationary case is independent of the viscoelastic properties [9,10], we will only recall here the main results of the linear analysis [1,2]. The Rayleigh number obtained as the eigenvalue of the linear part of system (7) is given by [2]:

$$Ra = \frac{(k^2 M_3 + \pi^2)(\pi^2 T_a + q^6)}{k^2\left[\pi^2(1 + k^2 M_3 (1 + M_1)\right]}, \tag{8}$$

where $q^2 = k^2 + \pi^2$ is the augmented dimensionless wavenumber. The minimum of the marginal curve ($\partial_k Ra = 0$) gives the critical wavenumber k_c and the associated critical Rayleigh number, Ra_c. In the present case the threshold value increases monotonically when T_a increases, therefore the rotation of the liquid layer has a stabilizing effect on the convection threshold. In addition, one observes from Eq. (8) that the threshold value decreases for an increase of either M_1 or M_3. This indicates the destabilizing effect of a magnetic field on the convection threshold.

A nonlinear analysis is needed to determine the type of convective motion which is expected to develop beyond the linear instability threshold. The study of the evolution of the convective pattern can be done by means of a multiple scale analysis [4]. We will assume that a convective cell of small amplitude is imposed on the basic flow. For values of the control parameter Ra, close to its threshold value Ra_c, the bifurcation parameter will be $\epsilon^2 = (Ra - Ra_c)/Ra_c$. We expand all functions in terms of ϵ and assume that all variations of the linearized solutions can be incorporated into a single amplitude function A. If this amplitude is of size ϵ (i.e. $O(\epsilon)$) then the interaction of the convective cell with itself forces a second harmonic and a mean state of correction of size $O\left(\epsilon^2\right)$, which in turn drives an $O\left(\epsilon^3\right)$ correction to the fundamental component of the imposed roll. A solvability criterion for this last correction yields an equation for the complex amplitude $A(X,T)$ of the imposed disturbance. This is a Ginzburg-Landau (GL) type equation. In the case of free-free boundary conditions, the z-dependence is contained entirely in the sine and cosine functions. Therefore, our expansion is $\mathbf{u} \to \epsilon\left(\mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \Theta\left(\epsilon^3\right)\right)$, and consequently $\boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}}_0 + \epsilon\boldsymbol{\mathcal{L}}_1 + \epsilon^2\boldsymbol{\mathcal{L}}_2 + \Theta\left(\epsilon^3\right)$ and $\boldsymbol{\mathcal{N}} = \epsilon\boldsymbol{\mathcal{N}}_0 + \epsilon^2\boldsymbol{\mathcal{N}}_1 + \Theta\left(\epsilon^3\right)$, where the expansions in the derivatives are $\partial_x \to \partial_x + \epsilon\partial_X$ and $\partial_t \to \epsilon^2\partial_T$, because A is a function of the slow time scale $T = \epsilon^2 t$ and the slow spatial scale $X = \epsilon x$. Inserting these expansions in Eq. (7), for each power of ϵ, one obtains a hierarchy of equations: $\boldsymbol{\mathcal{L}}_0\mathbf{u}_0 = 0$, $\boldsymbol{\mathcal{L}}_0\mathbf{u}_1 + \boldsymbol{\mathcal{L}}_1\mathbf{u}_0 = \boldsymbol{\mathcal{N}}_0$ and $\boldsymbol{\mathcal{L}}_0\mathbf{u}_2 + \boldsymbol{\mathcal{L}}_1\mathbf{u}_1 + \boldsymbol{\mathcal{L}}_2\mathbf{u}_0 = \boldsymbol{\mathcal{N}}_1$. These relationships must be solved subsequently and at each order one has to fulfill the solvability condition $\left\langle \mathbf{u}_0^\dagger \mid r.h.s. \right\rangle = \mathbf{0}$, where $\mathbf{u}_0^\dagger$ is the solution of the linear adjoint problem $\left(\boldsymbol{\mathcal{L}}^+\mathbf{u}^\dagger = 0\right)$. The notation $r.h.s$ is for the corresponding right hand side of the perturbation and $< \circ >$ denotes the inner product which is defined as a suitable volume integration. The solvability condition at $O\left(\epsilon^3\right)$ leads to an equation for the amplitude A that is written as

$$\tau_0 \frac{\partial A}{\partial T} = \xi_0^2 \frac{\partial^2 A}{\partial X^2} + \epsilon^2 A - g\left|A\right|^2 A\,. \tag{9}$$

Equation (9) is the GL equation and describes the variation on the slow time and spatial scales of the convective pattern. The coefficients τ_0 and ξ_0^2 are the growth rate of the amplitude and the curvature of the marginal stability curve, respectively; and they can be calculated straightforwardly from the linear theory analysis [10]. In Eq. (9), g is known as the nonlinear coefficient and for $g > 0$ we get a forward bifurcation (supercritical bifurcation), for $g < 0$ we get a backward bifurcation (subcritical bifurcation), and at $g = 0$ we get tricritical bifurcation point, which is the transition point between a subcritical and a supercritical bifurcation. The coefficients of this equation can be calculated using a standard procedure well detailed in Ref. [4]. For idealized boundary conditions after straightforward calculations, the explicit expressions of these coefficients can be written as

$$\tau_0 = \frac{(1+P)q^6 + (-1+P)\pi^2 T_a}{Pq^2\left(q^6 + \pi^2 T_a\right)} + \frac{\Gamma\left(q^6 - \pi^2 T_a\right)(-1+\Lambda)}{q^6 + \pi^2 T_a}\,, \tag{10}$$

$$\xi_0^2 = \frac{2q_c^4}{q_c^6 + \pi^2 T_a} - \frac{M_1 M_3 \pi^2}{(M_3 k_c^2 + \pi^2)\left((1+M_1)M_3 k_c^2 + \pi^2\right)}\,, \tag{11}$$

$$\begin{aligned} g =& \frac{\pi^2 q_c^2 \left(M_3(1+2M_1)k_c^2 + \pi^2\right)}{2\left(q_c^2(1+M_1)M_3 + \pi^2\right)} - \frac{\pi^2 q_c^2 T_a}{2P^2 k_c (q_c^6 + \pi^2 T_a)} \\ &+ \frac{a^2(1-\Lambda)\Gamma^2}{q_c^4}\left(9q_c^{16} + 8\pi^2 k_c^2(\pi^2 - q_c^2)^4 + 144\pi^4 k_c^4\right) \\ &- \frac{(1-\Lambda)\Gamma^2}{q_c^4}\left(9q_c^8(\pi^2 - q_c^2)^4 + 4\pi^2 k_c^2 q_c^8 + 64\pi^4 k_c^4\right)\,. \end{aligned} \tag{12}$$

It is interesting to note that, if we set $\Lambda = 1$ and $\Gamma = 0$, Eq. (12) reduces to the rotating Newtonian liquid case and we retrieve the coefficients found in reference [3]. Another limiting case is given when we set $T_a = 0$ in Eq. (12), where the viscoelastic limit described in reference [10] is retrieved.

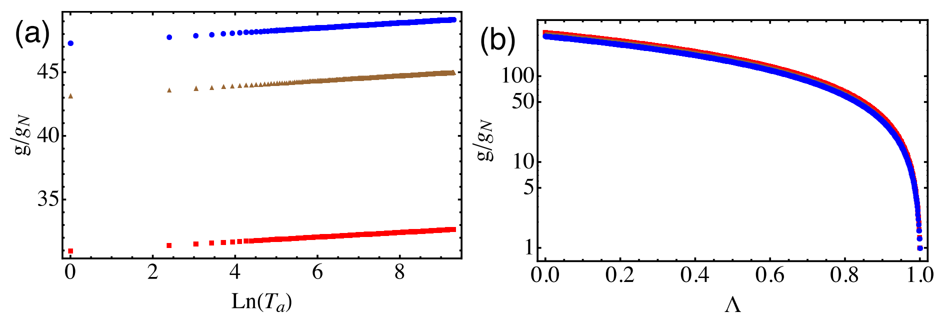

Figure 1: (Color online) Normalized cubic coefficient term, g/g_N, as a function of T_a (left) and Λ (right), for different values of M_1. In both cases the fixed parameters are $M_3 = 1.1$, $P = 7$, $\Gamma = 0.01$ and $a = 1$. Also, we have set in the left graph $\Lambda = 0.9$ and in the right graph $T_a = 10$. The different values of M_1 are represented by different symbols $M_1 = \{10, 5, 1\} = \{\blacksquare, \blacktriangle, \bullet\}$.

The main results are displayed in Fig.1, where we have plotted the normalized nonlinear coefficient as a function of the different parameters. Note that the normalization is done with respect to the case of a Newtonian fluid. Figure 1 shows g/g_N as a function of T_a and Λ for three realistic values of M_1. From Fig.1($left$), we see that g/g_N increases logarithmically with T_a and decreases when M_1 is increased. From Fig.1($right$), we see that g/g_N decreases when the ratio between retardation and relaxation times, Λ, increases. For the selected fixed parameters, one sees that the decaying behavior in Fig.1($right$) is almost independent of the magnetic field ($M_1 \sim H^2$). In the Maxwell case ($\Lambda = 0$) g/g_N reaches its maximum value. From the present results, we therefore conclude that there is a competition between the rotation, the magnetic and the viscoelastic effects. In fact, due to this competition a supercritical-subcritical transition (i.e. $g = 0$) can occur for some parameter values. Typically, without rotation, in viscoelastic magnetic fluids it occurs for models like the corotational Jeffrey's model because in this case the parameter a is close to zero [9].

Final Remarks. In this paper, we have presented the derivation of the amplitude equation of the convective roll patterns that arise in a rotating viscoelastic magnetic fluid layer heated from below. The viscoelastic properties are modeled through the Oldroyd constitutive equation. In the present paper, we focus on the stationary bifurcation that is more commonly observed in ferrofluids [14]. The kinetic coefficients of the Ginzburg-Landau equation have been calculated analytically. The rotation of the layer has a stabilizing effect while the magnetic field is destabilizing. The nonlinear coefficient term g increases logarithmically as a function of the rotation rate and it decreases for strong magnetic fields. The determination of the amplitude equation in the case of an oscillatory instability and in the case of a co-dimension two bifurcation is still in progress and will be presented elsewhere.

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