J. Math. Anal. Appl. 448 (2017) 968–981

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Duality properties of bounded torsion topological abelian groups

M.J. Chasco [a], X. Domínguez [b,*], M. Tkachenko [c]

[a] Departamento de Física y Matemática Aplicada, University of Navarra, Spain
[b] Departamento de Métodos Matemáticos y de Representación, Universidade da Coruña, Spain
[c] Departamento de Matemáticas, Universidad Autónoma Metropolitana, Mexico D.F., Mexico

ARTICLE INFO

Article history:
Received 29 March 2016
Available online 16 November 2016
Submitted by J. Bonet

Keywords:
Precompact group
Baire property
Pseudocompact
Countably compact
Pointwise convergence topology

ABSTRACT

Let G be a precompact, bounded torsion abelian group and $G_p^\wedge$ its dual group endowed with the topology of pointwise convergence. We prove that if G is Baire (resp., pseudocompact), then all compact (resp., countably compact) subsets of $G_p^\wedge$ are finite. We also prove that G is pseudocompact if and only if all countable subgroups of $G_p^\wedge$ are closed. We present other characterizations of pseudocompactness and the Baire property of $G_p^\wedge$ in terms of properties that express in different ways the abundance of continuous characters of G. Besides, we give an example of a precompact boolean group G with the Baire property such that the dual group $G_p^\wedge$ contains an infinite countably compact subspace without isolated points.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction

Pontryagin–van Kampen duality theorem is a deep result with far-reaching consequences in the theory of locally compact abelian groups. It allows the characterization of different properties of a locally compact group in terms of properties of the dual group. If the group is compact, its dual group is discrete, therefore the topological properties of the group are reflected by algebraic properties of the dual group.

In the non-locally compact case, much remains to be explored, although there have been important insights during the last years. For instance, the existence of precompact, noncompact Pontryagin reflexive groups was established in [1,12] and [6]. Once Comfort and Ross proved in [9] that the topology of a precompact group was the one induced by its continuous characters, a natural notion of duality in the class of precompact groups emerged. The dual group of a topological abelian group G is in this context $G_p^\wedge$, which means the group $G^\wedge$ of continuous characters of G endowed with the topology of pointwise convergence. Actually Raczkowski and Trigos-Arrieta proved in [18] that every precompact abelian group G

* Corresponding author.
E-mail addresses: mjchasco@unav.es (M.J. Chasco), xabier.dominguez@udc.es (X. Domínguez), mich@xanum.uam.mx (M. Tkachenko).

http://dx.doi.org/10.1016/j.jmaa.2016.11.029
0022-247X/© 2016 Elsevier Inc. All rights reserved.

is Comfort–Ross reflexive, i.e. the natural homomorphism from G to $(G_p^\wedge)_p^\wedge$ which takes each x to the evaluation $[\chi \in G_p^\wedge \mapsto \chi(x)]$ is a topological isomorphism.

There are many examples of properties of a precompact abelian group G with an equivalent counterpart in $G_p^\wedge$. For instance, G is compact if and only if $G_p^\wedge$ carries its maximal precompact topology, and G is pseudocompact if and only if all countable subgroups of $G_p^\wedge$ carry their maximal precompact topology [13].

This article is aimed at presenting further results along these lines for the class of precompact, bounded torsion abelian groups G. Our choice of bounded torsion groups is not accidental: the Pontryagin duality theory acquires several specific features in this class (see [4]). For instance the above pseudocompactness criterion can be reformulated for such groups G in the following way (Theorem 2.6): G is pseudocompact if and only if all countable subgroups of $G_p^\wedge$ are closed.

Also, Comfort–Ross duals of pseudocompact abelian groups are known examples of precompact groups without infinite compact subsets [1]. Similarly, some of the main results of this article, which we collect in the table below, can be summarized in this way: "If G is a precompact, bounded torsion abelian group with property (P) then the only subsets of $G_p^\wedge$ with property (Q) are the finite ones", where (P) and (Q) are two of the following topological properties, listed in decreasing order of generality: Baire, pseudocompact, countably compact, compact.

	(P)	(Q)
Theorem 3.3	Baire	Compact
Corollary 2.8	Pseudocompact	Countably compact
Proposition 2.10	Countably compact	Pseudocompact

Actually, Proposition 2.10 holds for all, not necessarily bounded torsion, precompact abelian groups. It is not clear whether the same restriction can be dropped in the remaining two cases. We also discuss the topological sharpness of these results. In particular we present in Example 3.12 a precompact boolean group G with the Baire property such that the dual group $G_p^\wedge$ contains an infinite countably compact subset.

Furthermore, in Proposition 2.4, Theorem 2.6 and Theorem 3.2 we give characterizations of the Baire property and pseudocompactness of $G_p^\wedge$ in terms of properties that express in different ways the abundance of continuous characters of G. Two different characterizations of pseudocompact bounded torsion abelian groups are given in the first place, which are obtained from known results; afterwards we study the Baire property whose characterization needs more work.

1.1. Notation, terminology, and preliminary facts

All groups we consider are assumed to be abelian. If every nonzero element of a group G has finite order, we say that G is a *torsion* group. If there exists a positive integer m with $mG = \{0\}$ we say that G is *bounded torsion*; the minimal m with this property is called the *period* of G.

As usual, ω is the set of natural numbers, $\mathbb{Z}$ stands for the set of integers and $\mathbb{R}$ is the set of real numbers. The quotient group $\mathbb{R}/\mathbb{Z}$ is denoted by $\mathbb{T}$. We will use the same notation for the elements of $\mathbb{T}$ and their representatives in the interval $(-1/2, 1/2]$.

A subset $\{x_1, \ldots, x_k\}$ of a group G is *independent* if the equality $n_1x_1 + \cdots + n_kx_k = 0$ with $n_1, \ldots, n_k \in \mathbb{Z}$ implies that $n_1x_1 = \cdots = n_kx_k = 0$. An infinite set $X \subset G$ is *independent* if every finite subset of X is independent. For every $n \in \mathbb{N}$ we put $G[n] := \{x \in G : nx = 0\}$. Clearly $G[n]$ is a subgroup of G. Furthermore, if G is a topological group, then $G[n]$ is a closed subgroup of G.

A topological group G is *precompact* if for every neighborhood U of 0 in G, finitely many translates of U cover G. In what follows we will use "precompact" as an abbreviation of "precompact and Hausdorff".

A Tychonoff space is said to be *pseudocompact* if every continuous real-valued function defined on it is bounded. Any Hausdorff pseudocompact group is precompact; actually a precompact group G is pseudocompact if and only if it meets each nonempty G_δ-subset of its Weil completion [10].

A subset B of a topological space X is said to be *nowhere dense* if the closure of B in X has no interior points. A space X has the *Baire property* if the intersection of every countable family of open dense subsets of X is dense in X or, equivalently, if the only open subset of X representable as a countable union of nowhere dense subsets of X is the empty set. It is a consequence of the Banach Category Theorem that a topological group G has the Baire property if and only if the intersection of any countable family of open dense subsets of G is nonempty or, equivalently, the group G is not a countable union of its nowhere dense subsets.

A *character* of a topological group G is a continuous homomorphism of G to $\mathbb{T}$. A topological group G is *MAP* (abbreviation for Maximal Almost Periodic) if for every $x \in G \setminus \{0\}$, there exists a character χ of G such that $\chi(x) \neq 0$. In other words, G is MAP if and only the characters of G separate points of G.

Given a topological group G, we denote by $G^\wedge$ the group of all characters of G, with pointwise addition. The symbol $\sigma(G, G^\wedge)$ denotes the *Bohr topology* of G, that is, the initial topology on G with respect to its continuous characters. We also denote by $G^\wedge_p$ the group $G^\wedge$ endowed with the pointwise convergence topology.

If G is bounded torsion, so is $G^\wedge$; in particular all characters of G have finite range, and the sets of the form $\Delta^\perp = \{\chi \in G^\wedge : \chi(\Delta) = \{0\}\}$, with Δ running over all finite subsets of G, form a basis of neighborhoods of 0 for the topology of $G^\wedge_p$.

A subgroup N of a topological group G is said to be *dually embedded* in G if every character of N extends to a character of G. Every subgroup of a precompact group is dually embedded. Every finite subgroup of a MAP group is dually embedded as well.

A nonempty subset A of a topological group G is said to be *quasi-convex* if for every $x \notin A$ there exists a character ψ of G such that $\psi(A)$ is contained in $[-1/4, 1/4]$ and $\psi(x) \notin [-1/4, 1/4]$. The topological group is said to be *locally quasi-convex* if it has a neighborhood base at 0 consisting of quasi-convex subsets. Every precompact group is locally quasi-convex. Every locally quasi-convex, Hausdorff group is MAP (see [3]).

A topological group is said to be *Pontryagin reflexive* if the natural evaluation mapping $\alpha_G : G \to (G^\wedge_{\text{co}})^\wedge_{\text{co}}$ is a topological isomorphism, where the subscript "co" stands for the compact-open topology on both groups $G^\wedge$ and $(G^\wedge_{\text{co}})^\wedge$. Pontryagin–van Kampen classical duality theorem asserts that all locally compact abelian groups are reflexive. The reader can find more information on recent developments in the Pontryagin reflexivity of abelian topological groups in the survey article [7].

2. Pseudocompactness of $G^\wedge_p$ for a bounded torsion group G

In the sequel we will need the following results:

Proposition 2.1. *Let G be a MAP group.*

(a) (See [9, Theorem 1.2].) *The group G is precompact if and only if its topology coincides with $\sigma(G, G^\wedge)$.*
(b) (See [18, Theorem 3.1: The Comfort–Ross duality].) *If G is precompact, the canonical homomorphism $G \to (G^\wedge_p)^\wedge_p$ is a topological isomorphism.*

It is worth noting that if the compact subsets of G and $G^\wedge_{\text{co}}$ are finite, then the Pontryagin duality and the Comfort–Ross duality coincide.

Following [20,1] we say that a subgroup D of a topological Abelian group G is *h-embedded* in G if every (not necessarily continuous) homomorphism $f : D \to \mathbb{T}$ can be extended to a *continuous* character $\tilde{f} : G \to \mathbb{T}$. It is clear that if D is h-embedded in G, then every homomorphism of D to $\mathbb{T}$ is continuous.

Proposition 2.2 *(See Proposition 2.1 in [1]). If every countable subgroup of a topological group G is h-embedded, then the countable subgroups of G are closed and the compact subsets of G are finite.*

Proposition 2.3 *(See Proposition 3.4 in [13]). A precompact group G is pseudocompact if and only if all countable subgroups of $G_p^\wedge$ are h-embedded.*

In the next proposition we characterize the MAP bounded torsion groups G with the pseudocompact dual $G_p^\wedge$.

Proposition 2.4. *Let G be a bounded torsion, MAP topological group. The following conditions are equivalent:*

(i) *$G_p^\wedge$ is pseudocompact.*
(ii) *For every sequence $\{g_n : n \in \omega\}$ of independent elements of G and every sequence $\{t_n : n \in \omega\}$ of elements of $\mathbb{T}$ such that the order of t_n divides that of g_n for every $n \in \omega$, there exists $\chi \in G^\wedge$ with $\chi(g_n) = t_n$ for every $n \in \omega$.*
(iii) *For every sequence $\{\Delta_n : n \in \omega\}$ of finite subsets of G, where $\langle \Delta_n \rangle \cap \langle \bigcup_{k<n} \Delta_k \rangle = \{0\}$ for every $n \in \omega$, and every sequence $\{\chi_n\}$ in $G^\wedge$, one has $\bigcap_{n\in\omega}(\chi_n + \Delta_n{}^\perp) \neq \emptyset$.*

Proof. Since all the above statements (i)–(iii) either hold or fail to hold simultaneously for G and $(G, \sigma(G, G^\wedge))$, we may assume that G is precompact.

(i) ⇒ (iii): Pick sequences $\{\Delta_n : n \in \omega\}$ and $\{\chi_n : n \in \omega\}$ as in (iii). Put $D = \bigoplus_{n\in\omega}\langle \Delta_n \rangle$. It is clear that the homomorphism $f : D \to \mathbb{T}$ given by $f(x) = \chi_n(x)$ for all $n \in \omega$ and $x \in \Delta_n$ is well defined. Proposition 2.3 applied to $G_p^\wedge$ and the Comfort–Ross duality together imply that the countable subgroup D is h-embedded in G. Hence there exists $\chi \in G^\wedge$ which coincides with f on D. Clearly $\chi \in \bigcap_{n\in\omega}(\chi_n + \Delta_n{}^\perp)$.

(iii) ⇒ (ii): Fix sequences $\{g_n\}$ and $\{t_n\}$ as in (ii). For every $n \in \omega$ the character κ_n of the finite group $\langle\{g_n\}\rangle$ defined by $\kappa_n(g_n) = t_n$ can be extended to a character χ_n of the precompact group G. By (iii), there is some $\chi \in G^\wedge$ such that $\chi \in \bigcap_{n\in\omega}(\chi_n + \{g_n\}^\perp)$, which means that $\chi(g_n) = t_n$ for every $n \in \omega$.

(ii) ⇒ (i): By Proposition 2.3 applied to $G_p^\wedge$ and the Comfort–Ross duality, it suffices to show that all countable subgroups of G are h-embedded. Consider a countable subgroup C of G. We may assume that C is infinite. Take any homomorphism $f : C \to \mathbb{T}$. Let us show that there exists $\chi \in G^\wedge$ which coincides with f on C. We can express the countably infinite, bounded torsion group C as $C = \bigoplus_{n\in\omega}\langle g_n \rangle$, where $\{g_n : n \in \omega\} \subset G$ is an independent sequence [19, 4.3.5]. Since the order of $f(g_n)$ clearly divides that of g_n for every n, we conclude that there exists $\chi \in G^\wedge$ such that $\chi(g_n) = f(g_n)$ for every n. That is, χ and f coincide on C. □

Our next aim is to prove that in the case of bounded torsion groups the sufficient condition for pseudocompactness given in Proposition 2.3 can be replaced by a weaker one. We start with a lemma.

Lemma 2.5. *Let G be a bounded torsion topological group. The following properties are equivalent:*

(i) *All subgroups of G are closed.*
(ii) *Every homomorphism of G to $\mathbb{T}$ is continuous.*

Proof. Let us show that (i) ⇒ (ii). If $f : G \to \mathbb{T}$ is a homomorphism, then $f(G) \subset \mathbb{T}[n]$, where n is the period of G. Hence the kernel of f, say, K is a subgroup of G which has finite index in G. By the assumptions of the lemma, K is closed in G. Hence G is a disjoint union of finitely many closed cosets of K. This implies in turn that each of these cosets is open in G. Therefore K is an open subgroup of G and the homomorphism f is continuous.

The implication (ii) ⇒ (i) is known and true without assuming that G is bounded torsion; we give the argument for the sake of completeness. Consider a subgroup H of G and fix $x \in G \setminus H$. Let $\varphi : G \to G/H$ be the corresponding quotient map. Since $\varphi(x) \neq 0$ there exists a homomorphism $\chi : G/H \to \mathbb{T}$ with $\chi(\varphi(x)) \neq 0$. The homomorphism $\chi \circ \varphi : G \to \mathbb{T}$ is continuous by hypothesis and separates x from H. □

Theorem 2.6. *Let G be a MAP, bounded torsion group. The following conditions are equivalent:*

(a) *All countable subgroups of G are $\sigma(G, G^\wedge)$-closed.*
(b) *The group $G_p^\wedge$ is pseudocompact.*

In addition, if G is locally quasi-convex, both (a) *and* (b) *are equivalent to*

(a′) *All countable subgroups of G are closed.*

Proof. We will use the following observation: If $H = (G, \sigma(G, G^\wedge))$, then by Proposition 2.3 applied to $G_p^\wedge$ and the Comfort–Ross duality, we obtain that the group $G_p^\wedge = H_p^\wedge$ is pseudocompact if and only if every countable subgroup of H is h-embedded.

(a) ⇒ (b): Take a countable subgroup S of H. By hypothesis, every subgroup of S is closed in H, hence in S. By Lemma 2.5, every homomorphism from S to $\mathbb{T}$ is continuous. Since H is precompact, every subgroup of H is dually embedded. We conclude that S is h-embedded in H.

(b) ⇒ (a): By hypothesis, all countable subgroups of H are h-embedded. By Proposition 2.2, they are closed in H as well.

It is clear that (a) ⇒ (a′), even without the local quasi-convexity assumption. Let us prove (a′) ⇒ (a). Since G is a bounded torsion, locally quasi-convex group, it has a basis of neighborhoods at 0 formed by subgroups (see [4, Proposition 2.1]), therefore it can be embedded into a product of discrete groups. This implies in turn that G is a nuclear group, and in particular every closed subgroup of G is $\sigma(G, G^\wedge)$-closed (see [5, Corollary 8.6]). □

Remark 2.7. It turns out that Theorem 2.6 cannot be extended to precompact torsion groups, neither to precompact torsion free groups (notice that all precompact groups are locally quasi-convex).

Indeed, let $Q_q = \{t \in \mathbb{T} : q^k t = 0 \text{ for some } k \in \mathbb{N}\}$ be the quasicyclic subgroup of $\mathbb{T}$, where q is a prime number. We consider Q_q with the topology inherited from $\mathbb{T}$. Clearly Q_q is a precompact torsion group. Every proper subgroup of Q_q is finite and hence is closed in Q_q. However $(Q_q)_p^\wedge$ is not pseudocompact. Indeed, it is countable, infinite, non-discrete, and has a countable base. Hence this group is *homeomorphic* to the space of rational numbers endowed with the usual linear order topology.

Consider also the infinite cyclic group $\mathbb{Z}$ endowed with the precompact topological group topology τ whose base at zero consists of the subgroups $n\mathbb{Z}$, where $n \geq 1$ is an integer. Clearly every non-trivial subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n > 1$, so all subgroups of $G = (\mathbb{Z}, \tau)$ are closed. Once again, $G_p^\wedge$ is countably infinite and non-discrete, so it is not pseudocompact.

Corollary 2.8. *Let G be a pseudocompact, bounded torsion group. Then every countably compact subset of $G_p^\wedge$ is finite.*

Proof. By Theorem 2.6 and the Comfort–Ross duality, all countable subgroups of $G_p^\wedge$ are closed. Hence $G_p^\wedge$ is a bounded torsion abelian group all whose countable subgroups are closed. By Theorem 2.1 in [22], every countably compact subset of $G_p^\wedge$ is finite. □

We will see in Example 3.12 that one cannot weaken "pseudocompact" to "Baire" in Corollary 2.8. However, if G is a Baire bounded torsion group, every *compact* subset of $G_p^\wedge$ is finite (Theorem 3.3).

Proposition 2.10 below shows that one can interchange "pseudocompact" and "countably compact" in Corollary 2.8, even without additional assumptions on the algebraic structure of the group G. First we recall a useful topological concept. A subset B of a Tychonoff space X is said to be *bounded* in X if every continuous real-valued function on X is bounded on B. It is clear that every pseudocompact subspace of a Tychonoff space is bounded.

The following result is a special case of [13, Theorem 4.2] which generalizes Grothendieck's theorem about compact subsets of $C_p(X)$; we apply it in the proof of Proposition 2.10.

Theorem 2.9. *Let X be a countably compact space and Y a metrizable space. Then the closure of every bounded subset of $C_p(X, Y)$ is compact.*

Proposition 2.10. *Let G be a countably compact topological group. Then every bounded subset of $G_p^\wedge$ is finite. In particular all pseudocompact subspaces of $G_p^\wedge$ are finite as well.*

Proof. Let P be a bounded subset of $G_p^\wedge$. Then the closure of P in $G_p^\wedge$, say, K is also a bounded subset of $G_p^\wedge$. Denote by $C_p(G, \mathbb{T})$ the space of continuous functions on G with values in $\mathbb{T}$ endowed with the pointwise convergence topology. Then $G_p^\wedge$ is clearly a closed subspace of $C_p(G, \mathbb{T})$, so K is a closed bounded subset of $C_p(G, \mathbb{T})$. Since G is countably compact and $\mathbb{T}$ is metrizable, it follows from Theorem 2.9 that K is compact. Note that every countably compact group is pseudocompact, so Proposition 2.3 implies that all countable subgroups of $G_p^\wedge$ are h-embedded. Hence all compact subsets of $G_p^\wedge$ are finite according to Proposition 2.2. We conclude therefore that the set K and its subset P are finite. □

The particularization of Proposition 2.10 to the case of a compact group G gives the following result established by Trigos-Arrieta in [23, Theorem 4.4] (see also [2, Theorem 9.9.42]):

Corollary 2.11. *Let H be an abelian group and τ the finest precompact topology on H. Then every bounded subset of (H, τ) is finite.*

Remark 2.12. It is natural to ask whether one can strengthen Corollary 2.8 and/or Proposition 2.10 by replacing "countably compact" to "pseudocompact". In other words, we wonder whether pseudocompact subspaces of $G_p^\wedge$, for a pseudocompact bounded torsion group G, are finite. It turns out that the answer in both cases is "No", so both Corollary 2.8 and Proposition 2.10 are quite sharp. Indeed, it is shown in [21, Theorem 3.3] that there exists an infinite pseudocompact boolean group G such that the dual group $G_p^\wedge$ is topologically isomorphic to G and, hence, pseudocompact. In fact, the cardinality of such a group G can be arbitrary big.

3. The Baire property on $G_p^\wedge$ for a bounded torsion group G

Our next goal is to give a characterization of the Baire property for bounded torsion groups similar in spirit to the one obtained for pseudocompactness in Proposition 2.4. We start with a lemma:

Lemma 3.1. *Let G be a MAP bounded torsion topological group. For every nowhere dense subset F of $G_p^\wedge$ and every finite subgroup $K \subset G$, one can find a finite subgroup L of G and $\chi \in G_p^\wedge$ such that $K \cap L = \{0\}$ and $(\chi + L^\perp) \cap F = \emptyset$.*

Proof. Let F be a nowhere dense subset of $G_p^\wedge$. We can assume that G is infinite — otherwise the dual group $G_p^\wedge \cong G$ is finite and discrete, so the set F must be empty. Let $n \geq 2$ be the period of G. Then $\chi(G) \subset \mathbb{T}[n]$, for every character $\chi \in G_p^\wedge$.

The bounded torsion group G is algebraically isomorphic with a direct sum of finite cyclic groups whose orders divide n [19, 4.3.5]. Thus $G = \bigoplus_{z \in Z} \langle z \rangle$ algebraically, where Z is an independent subset of G. Let us denote by G_d the group G endowed with the discrete topology. It is clear that $G^\wedge$ is a subgroup of $(G_d)^\wedge$. Actually, since G is a MAP group, $G_p^\wedge$ is a dense subgroup of the compact group $(G_d)_{\mathrm{co}}^\wedge$; this follows from Theorem 1.9 in [9]. Besides, note that $(G_d)_{\mathrm{co}}^\wedge$ is naturally topologically isomorphic with $\Pi_Z = \prod_{z \in Z} \mathbb{T}[n_z]$, where n_z is the order of z, via the canonical isomorphisms $\langle z \rangle^\wedge \cong \mathbb{T}[n_z]$; in what follows we make implicit use of this identification. In particular we can regard F as a nowhere dense subset of Π_Z. Also, since the closure of a nowhere dense set is nowhere dense, we can assume that F is closed in Π_Z.

For every $B \subset Z$ define $\Pi_B := \prod_{z\in B} \mathbb{T}[n_z]$, and let $\pi_B\colon \Pi_Z \to \Pi_B$ be the projection. Clearly $\pi_B(G_p^\wedge)$ is a dense subgroup of Π_B for every $B \subset Z$. There exists a finite subset X of Z such that $K \subset \langle X\rangle$. Let us put $Y = Z \setminus X$.

Let us prove that $\pi_Y(F)$ is nowhere dense in Π_Y. For every $s \in \Pi_X$, let

$$P(s) = \{z \in F : z(x) = s(x) \text{ for each } x \in X\}.$$

It is easy to see that for every $s \in \Pi_X$ the set $\pi_Y(P(s))$ is nowhere dense in Π_Y. Indeed, since $P(s)$ is closed in Π_Z and π_Y is a closed map, $\pi_Y(P(s))$ is also closed in Π_Y, so if $\pi_Y(P(s))$ fails to be nowhere dense, it must contain a non-empty open set in Π_Y, say, O. Then

$$O \times \{s\} \subset \pi_Y(P(s)) \times \{s\} = P(s) \subset F.$$

Since Y is cofinite in Z, $O \times \{s\}$ is a non-empty open set in Π_Z, which contradicts the fact that F is nowhere dense in Π_Z. Therefore the sets $\pi_Y(P(s))$, with $s \in \Pi_X$, are nowhere dense in Π_Y. Notice that $F = \bigcup\{P(s) : s \in \Pi_X\}$. Hence the set $\pi_Y(F)$ is the union of the finite family $\{\pi_Y(P(s)) : s \in \Pi_X\}$ of closed nowhere dense sets in Π_Y. This clearly implies that $\pi_Y(F)$ is nowhere dense in Π_Y.

Since $\pi_Y(G_p^\wedge)$ is dense in Π_Y, we deduce that $\pi_Y(F)$ is nowhere dense in $\pi_Y(G_p^\wedge)$. In particular we can find a character $\chi_0 \in G^\wedge$ such that the intersection of some basic neighborhood of $\pi_Y(\chi_0)$ in Π_Y with $\pi_Y(G_p^\wedge)$ does not meet $\pi_Y(F)$. This means that there exists a finite subset $\Delta \subset Y$ such that no $\chi \in F$ coincides with χ_0 on Δ, i.e. $(\chi_0 + \Delta^\perp) \cap F = \emptyset$. Let $L = \langle\Delta\rangle$. Then $L \subset \langle Y\rangle$ and $K \subset \langle X\rangle$, so $K \cap L = \{0\}$. This completes the proof. □

In Theorem 3.2 below we characterize the MAP bounded torsion groups G such that the dual group $G_p^\wedge$ has the Baire property. This theorem is a natural but more complicated analogue of Proposition 2.4.

Theorem 3.2. *Let G be a MAP, bounded torsion topological group. The following conditions are equivalent:*

(i) *$G_p^\wedge$ has the Baire property.*
(ii) *For every sequence $\{\Delta_n : n \in \omega\}$ of finite subsets of G, where $\langle\Delta_n\rangle \cap \langle\Delta_k\rangle = \{0\}$ if $n \neq k$, and every sequence $\{\chi_n\}$ in $G_p^\wedge$, there exists an infinite set $I \subset \omega$ such that $\bigcap_{n\in I}(\chi_n + \Delta_n^\perp) \neq \emptyset$.*

Proof. Assume that the dual group $G_p^\wedge$ has the Baire property. Take an arbitrary sequence $\{(\Delta_n, \chi_n) : n \in \omega\}$ as in (ii). Our aim is to find an infinite set $I \subset \omega$ and a character $\chi \in G^\wedge$ such that $\chi(x) = \chi_n(x)$ for all $n \in I$ and $x \in \Delta_n$.

For every $n \in \omega$, put $U_n = \bigcup_{m>n}(\chi_m + \Delta_m^\perp)$. It is clear that the sets U_n are open in $G_p^\wedge$. Let us verify that U_n is dense in $G_p^\wedge$ for each $n \in \omega$. Take an arbitrary character $\chi_0 \in G_p^\wedge$, a finite set $C \subset G$ and consider the basic open set $\chi_0 + C^\perp$ in $G_p^\wedge$. Since the group $\langle C\rangle$ is finite and $\langle\Delta_k\rangle \cap \langle\Delta_l\rangle = \{0\}$ if $k \neq l$, there are at most finitely many indices $k \in \omega$ such that $\langle C\rangle \cap \langle\Delta_k\rangle \neq \{0\}$. Take an integer $m > n$ such that $\langle C\rangle \cap \langle\Delta_m\rangle = \{0\}$. The finite group $\langle C\rangle + \langle\Delta_m\rangle$ is dually embedded in the MAP group G, therefore there exists a character $\chi \in G_p^\wedge$ such that $\chi(x) = \chi_0(x)$ for each $x \in C$ and $\chi(x) = \chi_m(x)$ for each $x \in \Delta_m$. It is clear that $\chi \in (\chi_0 + C^\perp) \cap U_n \neq \emptyset$, which implies that U_n is dense in $G_p^\wedge$.

The group $G_p^\wedge$ has the Baire property, hence the set $S = \bigcap_{n\in\omega} U_n$ is non-empty. Take an element $\chi \in S$. It follows from our choice of χ that for each $n \in \omega$, there exists $m > n$ such that $\chi(x) = \chi_m(x)$ for all $x \in \Delta_m$. In other words, the set

$$I = \{m \in \omega : \chi(x) = \chi_m(x) \text{ for each } x \in \Delta_m\}$$

is infinite. This proves the necessity.

To prove the sufficiency, assume the group G satisfies (ii). Suppose for a contradiction that the group $G_p^\wedge$ is not Baire. Then there exists an increasing sequence $\{F_n : n \in \omega\}$ of closed nowhere dense sets in $G_p^\wedge$ such that $G_p^\wedge = \bigcup_{n\in\omega} F_n$. Take a non-empty basic open set $\chi_0 + \Delta_0^\perp$ in $G_p^\wedge$ such that $(\chi_0 + \Delta_0^\perp) \cap F_0 = \emptyset$. Denote by K_0 the subgroup of G generated by Δ_0. Clearly K_0 is finite.

Assume that for some $n \in \omega$ we have defined $\chi_0, \dots, \chi_n \in G_p^\wedge$ and finite subsets $\Delta_0, \dots, \Delta_n$ of G such that $(\chi_k + \Delta_k^\perp) \cap F_k = \emptyset$ for each $k \le n$, and that the groups $\langle\Delta_k\rangle$ and $\langle\Delta_l\rangle$ with $0 \le k < l \le n$ have trivial intersections. Denote by K_n the subgroup of G generated by the set $\bigcup_{k\le n} \Delta_k$. Clearly K_n is finite. By Lemma 3.1, there exists a non-empty basic open set $\chi_{n+1} + \Delta_{n+1}^\perp$ in $G_p^\wedge$ such that $(\chi_{n+1} + \Delta_{n+1}^\perp) \cap F_{n+1} = \emptyset$ and the groups K_n and $\langle\Delta_{n+1}\rangle$ have trivial intersection. This finishes our construction of the sequence $\{(\chi_n, \Delta_n) : n \in \omega\}$. It follows from our construction that $\langle\Delta_n\rangle \cap \langle\Delta_k\rangle = \{0\}$ if $n \ne k$.

According to (ii), we can find an infinite subset I of ω and a character $\chi \in G^\wedge$ such that $\chi \in \chi_n + \Delta_n^\perp$ for all $n \in I$. This implies that $\chi \notin F_n$, for each $n \in I$. Since $F_n \subset F_{n+1}$ for each $n \in \omega$, we conclude that $\chi \notin \bigcup_{n\in\omega} F_n = G_p^\wedge$. This contradiction shows that G is Baire. □

Let us recall that a space X is *scattered* if every nonempty subspace Y of X contains an isolated point. It is known that a compact Hausdorff space admits a continuous mapping onto the closed unit interval $[0, 1]$ if and only if it is not scattered (see [15, Proposition 3.5] for a proof).

Theorem 3.3. *Let G be a precompact, Baire, bounded torsion group. Then every compact subset of $G_p^\wedge$ is finite.*

Proof. Suppose to the contrary that $G_p^\wedge$ contains an infinite compact subset. Let m be the minimal positive integer such that some infinite compact subset L_m of $G_p^\wedge$ is contained in $G_p^\wedge[m]$. Since $G_p^\wedge$ (as well as G) is a bounded torsion group, such an m exists.

By the assumptions of the theorem, G is precompact and has the Baire property. Therefore its dual group $G_p^\wedge$ does not contain any nontrivial convergent sequences (see [6, Corollary 2.4]). By Theorem 4 in [16], every infinite compact scattered space contains non-trivial convergent sequences. Therefore no infinite closed subset of the compact space L_m is scattered. Let L be a closed infinite subset of L_m which does not contain isolated points. As the set $L_m \cap G_p^\wedge[k]$ is finite for each positive integer $k < m$, there exists a non-empty open set U in L such that $L' = \mathrm{cl}_L U$ is disjoint from $G_p^\wedge[k]$, for each $k < m$. Then L' is also an infinite closed subset of L without isolated points and the order of every element of L' is equal to m.

Let $\varphi\colon L' \to [0, 1]$ be a continuous, onto mapping. Let also K be a closed subset of L' such that $\varphi(K) = [0, 1]$ and the restriction of φ to K is *irreducible*, that is, $\varphi(K') \ne [0, 1]$ for every proper closed subspace K' of K (see [11, 3.1.C]).

In the sequel we need the following simple facts:

Claim A. *For every finite subgroup P of $G_p^\wedge$, the set*

$$K_P = \{x \in K : |P \cap \langle x\rangle| \ge 2\}$$

is finite.

Indeed, for an element $y \in P$ and an integer i with $1 \le i < m$, let $K[i, y] = \{x \in K : ix = y\}$. It is clear that the set $K[i, y]$ is compact and $i(x - z) = 0$ for all $x, z \in K[i, y]$, whence it follows that $K[i, y] - K[i, y] \subset G^\wedge[i]$. Since the set $K[i, y] - K[i, y]$ is compact and $1 \le i < m$, our choice of m implies that $K[i, y]$ is finite. Hence the set $K_P \subset \bigcup\{K[i, y] : 1 \le i < m,\ y \in P\}$ is finite as well.

Claim B. *For every finite subgroup P of $G_p^\wedge$ and an arbitrary non-empty open interval $J \subset [0,1]$, there exists an element $y \in K$ such that $\langle y\rangle \cap P = \{0\}$ and $\varphi(y) \in J$.*

Indeed, choose $a, b \in J$ with $a < b$ and let $J^* = [a,b]$ and $K^* = K \cap \varphi^{-1}(J^*)$. Since $\varphi(K) = [0,1]$, K^* is an infinite closed (hence compact) subset of K. By Claim A, we can take an element $y \in K^* \setminus K_P$. Then clearly $\langle y\rangle \cap P = \{0\}$ and $\varphi(y) \in J^* \subset J$.

Claim C. *For every finite subgroup P of $G_p^\wedge$ and an arbitrary integer $n \geq 1$, there exists a finite independent set $B \subset K$ such that $\langle B\rangle \cap P = \{0\}$ and $\varphi(B)$ is a 2^{-n}-net for $[0,1]$ with respect to the usual metric in $[0,1]$.*

Let $N = 2^{n+1}$. We define the required set $B = \{x_i : 1 \leq i \leq N\} \subset K$ as follows. First we take a family $\{J_i : 1 \leq i \leq N\}$ of non-empty open intervals in $[0,1]$ such that every open interval in $[0,1]$ of length greater than or equal to 2^{-n} contains one of the intervals J_i. For example, a uniform partition of $[0,1]$ into subintervals of length 2^{-n-1} has this property. By Claim B, there exists an element $x_1 \in K$ such that $\langle x_1\rangle \cap P = \{0\}$ and $\varphi(x_1) \in J_1$. Assume that for some $k < N$ we have defined an independent subset $\{x_1, \ldots, x_k\}$ of K such that $P \cap \langle x_1, \ldots, x_k\rangle = \{0\}$ and $\varphi(x_i) \in J_i$ for each $i \leq k$. Then $S = P + \langle x_1, \ldots, x_k\rangle$ is a finite subgroup of $G^\wedge$, so Claim B implies that there exists $x_{k+1} \in K$ such that $\langle x_{k+1}\rangle \cap S = \{0\}$ and $\varphi(x_{k+1}) \in J_{k+1}$. It follows from our choice of x_{k+1} that the set $\{x_1, \ldots, x_k, x_{k+1}\}$ is independent and $P \cap \langle x_1, \ldots, x_{k+1}\rangle = \{0\}$.

At the N-th step we obtain the independent set $B = \{x_i : 1 \leq i \leq N\} \subset K$. It follows from our choice of the intervals J_i and points x_i with $\varphi(x_i) \in J_i$ that $\varphi(B)$ is a 2^{-n}-net for $[0,1]$. It is also clear from our construction that $\langle B\rangle \cap P = \{0\}$. This proves Claim C.

We turn back to the proof of the theorem. Making use of Claim C, one can easily construct by induction a sequence $\{\Delta_n = A_n \cup B_n : n \in \omega\}$ of finite subsets of K satisfying the following conditions for each $n \in \omega$:

(i) $A_n \cap B_n = \emptyset$;
(ii) $\Delta_n \cap \Delta_k = \emptyset$ if $k \neq n$;
(iii) both $\varphi(A_n)$ and $\varphi(B_n)$ are 2^{-n}-nets in $[0,1]$ with respect to the usual metric in $[0,1]$;
(iv) the set $\bigcup_{n\in\omega} \Delta_n$ is independent in $G^\wedge$.

For every $n \in \omega$, we define a subset U_n of G by letting

$$\begin{aligned} U_n &= \{g \in G : (\exists\, p > n)\, (\forall x \in A_p)\, (\forall y \in B_p)\, [x(g) = 0,\, y(g) = 1/m]\} \\ &= \bigcup_{p>n} \Big(\bigcap_{x\in A_p} x^{-1}(\{0\}) \cap \bigcap_{y\in B_p} y^{-1}(\{1/m\}) \Big). \end{aligned}$$

It is clear from the above definition that each U_n is open in G.

Let us verify that U_n is dense in G for every $n \in \omega$. Take an arbitrary element $g_0 \in G$ and let $V = g_0 + \{x_1, \ldots, x_k\}^\perp$ be a basic neighborhood of g_0 in G, where $x_j \in G_p^\wedge$ for $j = 1, \ldots, k$. We have to find an element $g \in U_n \cap V$. Take an integer $p > n$ such that $\langle x_1, \ldots, x_k\rangle \cap \langle \Delta_p\rangle = \{0\}$. This is possible due to (ii) and (iv).

Let C be the subgroup of $G_p^\wedge$ generated by the set $\Delta_p \cup \{x_1, \ldots, x_k\}$. Since the finite group C is dually embedded in the precompact group $G_p^\wedge$, the character κ on C defined by $\kappa(x_j) = x_j(g_0)$ for $j = 1, \ldots, k$, $\kappa(x) = 0$ for each $x \in A_p$, and $\kappa(y) = 1/m$ for each $y \in B_p$ can be extended to a character g of $G_p^\wedge$. We identify g with an element of G which belongs to $V \cap U_n$. So this set is nonempty and each U_n is dense in G.

By hypothesis the group G has the Baire property, therefore $\bigcap_{n\in\omega} U_n \neq \emptyset$. Take an element $g \in \bigcap_{n\in\omega} U_n$. Then there exists an infinite subset I of ω such that $g(A_p) = \{0\}$ and $g(B_p) = \{1/m\}$ for every $p \in I$ (we identify g with a character of $G_p^\wedge$). Let us see that both $\bigcup_{p\in I} A_p$ and $\bigcup_{p\in I} B_p$ are dense in K, which clearly contradicts the continuity of the character g. Indeed, since φ is a closed map, it follows from (iii) that $\varphi(\overline{\bigcup_{p\in I} A_p}) = \overline{\varphi(\bigcup_{p\in I} A_p)} = [0,1]$, and the same equality holds for the sets B_p in place of A_p. As φ is also irreducible, both sets $\bigcup_{p\in I} A_p$ and $\bigcup_{p\in I} B_p$ are dense in K, as claimed. This completes the proof of the theorem. □

Problem 3.4. Can one drop "bounded torsion" in the assumptions of Theorem 3.3? In other words, is it true that all compact subsets of $G_p^\wedge$ are finite provided that G is a precompact group with the Baire property?

Corollary 3.5. *Let G be a MAP bounded torsion group such that $G_p^\wedge$ is a Baire space. Then every $\sigma(G, G^\wedge)$-compact subset of G is finite.*

Proof. Apply Theorem 3.3 to the precompact, Baire, bounded torsion group $G_p^\wedge$. Note that according to Proposition 2.1, $(G, \sigma(G, G^\wedge)) \cong (G_p^\wedge)_p^\wedge$. □

Recently many non-compact reflexive groups have been found among precompact groups. It is shown in [1, Theorem 2.8] and [12, Theorem 6.1] that a pseudocompact group without infinite compact subsets is reflexive. A slightly more general fact is established in [6, Theorem 2.8]: Every precompact Baire group without infinite compact subsets is reflexive provided that it satisfies the so-called *Open Refinement Condition* (see [6, p. 2638]). The following Corollary implies all these results for bounded torsion groups.

Corollary 3.6. *Let G be a precompact bounded torsion group which is a Baire space without infinite compact subsets. Then G is Pontryagin reflexive.*

Proof. By Theorem 3.3, all compact subsets of $G_p^\wedge$ are finite. The same is true by hypothesis for the compact subsets of G, so Pontryagin duality of G coincides with Comfort–Ross duality and G is Pontryagin reflexive. □

A locally quasi-convex group G is *g-barrelled* if every compact subset of $G_p^\wedge$ is equicontinuous. The class of g-barrelled groups was introduced in [8]. It includes all locally quasi-convex groups that are Čech-complete, separable Baire, or pseudocompact.

Corollary 3.7. *Let G be a precompact, bounded torsion group which is a Baire space. Then G is g-barrelled and its topology is the only locally quasi-convex topology on the abelian group G whose group of characters is $G^\wedge$.*

Proof. By Theorem 3.3, all compact subsets of $G_p^\wedge$ are finite. This clearly implies that G is g-barrelled.

Any g-barrelled topological group topology τ on an Abelian group is the finest locally quasi-convex topology with the same group of characters as τ [8]. Further, by Proposition 2.1(a), any precompact topological group topology τ on an Abelian group is the coarsest (locally quasi-convex) topology with the same group of characters as τ. This completes the proof. □

Corollary 3.8. *Let G be a locally quasi-convex, bounded torsion group such that $G_p^\wedge$ is a Baire space. Then G is reflexive if and only if G is g-barrelled.*

Proof. Since the group G is locally quasi-convex, the canonical homomorphism $\alpha_G : G \to (G_{co}^\wedge)_{co}^\wedge$ is open and injective. By Corollary 3.5, all compact subsets of G are finite. Hence the compact-open topology and the pointwise convergence topology coincide on $G^\wedge$. This implies that the map α_G is onto.

Finally, α_G is continuous if and only if the compact subsets of $G^\wedge_{co} \cong G^\wedge_p$ are equicontinuous [7] if and only if G is g-barrelled. □

In the next problem we actually ask whether Theorem 3.3 characterizes the Baire property in precompact boolean groups:

Problem 3.9. Let G be a precompact boolean group such that every compact subset of the dual group $G^\wedge_p$ is finite. Does G have the Baire property?

It is natural to ask, after Theorem 3.3, whether all *countably compact* subsets of $G^\wedge_p$ are finite, for each bounded torsion group G with the Baire property. In Example 3.12 below we present a precompact boolean group G with the Baire property such that the dual group $G^\wedge_p$ contains a big countably compact subset, thus answering the question in the negative. Our construction of such a group G does not require extra set-theoretic assumptions and can be easily visualized modulo some facts from the C_p-theory. We start with the following two lemmas.

Lemma 3.10. *Let X be a Tychonoff space satisfying the following condition:*

(∗) *For every sequence $\{(\Delta_n, f_n) : n \in \omega\}$, where the sets Δ_n are finite, pairwise disjoint subsets of X and the functions $f_n\colon \Delta_n \to \mathbb{Z}(2)$ are arbitrary, one can find an infinite set $I \subset \omega$ and a continuous function $f\colon X \to \mathbb{Z}(2)$ such that f_n and the restriction of f to Δ_n coincide for each $n \in I$.*

Then the group $H = C_p(X, \mathbb{Z}(2))$ has the Baire property.

Proof. Let $\{F_n : n \in \omega\}$ be a sequence of closed nowhere dense sets in H. It suffices to show that $H \neq \bigcup_{n\in\omega} F_n$. We can assume without loss of generality that $F_n \subset F_{n+1}$, for each $n \in \omega$.

For a non-empty set $A \subset X$ and a function $g\colon A \to \mathbb{Z}(2)$, we put

$$W(A, g) = \{f \in H : f(x) = g(x) \text{ for each } x \in A\}.$$

The sets of the form $W(A, g)$ form a base for the topology of the group H. Arguing as in the proof of Lemma 3.1, one can verify that for every finite subset D of X and a closed nowhere dense set F in H, there exists a basic open set $W(A, g)$ in H such that $W(A, g) \cap F = \emptyset$ and $A \cap D = \emptyset$. Therefore, we can construct by induction a sequence $\{W(\Delta_n, g_n) : n \in \omega\}$ of non-empty basic open sets in H such that $W(\Delta_n, g_n) \cap F_n = \emptyset$ and $\Delta_n \cap \Delta_k = \emptyset$ whenever $k < n$. Since X satisfies condition (∗) of the lemma, we can find an infinite set $I \subset \omega$ and a function $g \in H$ such that g_n and g coincide on Δ_n, for each $n \in I$. Then $g \in \bigcap_{n\in I} W(\Delta_n, g_n)$ and, hence, $g \notin \bigcup_{n\in I} F_n = \bigcup_{n\in\omega} F_n$. We have thus proved that $H \neq \bigcup_{n\in\omega} F_n$, so the group H has the Baire property. □

The next fact was established by Pytkeev in [17]. Since this source can hardly be accessed, we supply the reader with a proof.

Lemma 3.11. *Let X be a Tychonoff space such that every countable subspace of X is scattered and C^*-embedded in X. Then every countable infinite family γ of pairwise disjoint finite subsets of X contains an infinite subfamily λ such that the set $\bigcup \lambda$ is discrete.*

Proof. Let $\{A_n : n \in \omega\}$ be a faithful enumeration of the family γ and $Y = \bigcup \gamma$. Then Y is countable and, hence, scattered. Therefore one can enumerate Y in type β for some $\beta < \omega_1$, say, $Y = \{y_\nu : \nu < \beta\}$ such that every initial segment $Y_\delta = \{y_\nu : \nu < \delta\}$ is open in Y.

We apply induction on the order type β of Y. If $\beta = \omega$, then Y is discrete and there is nothing to prove. Assume that the conclusion of the lemma is valid whenever the order type of Y is less than α, where $\omega < \alpha < \omega_1$.

Case 1. α is a limit ordinal. There exists a strictly increasing sequence $\{\beta_k : k \in \omega\}$ of infinite ordinals such that $\alpha = \sup_{k\in\omega} \beta_k$. For every $k \in \omega$, let $Z_k = Y_{\beta_k}$. Then Z_k is open in Y, the order type of Z_k is β_k, and $Y = \bigcup_{k\in\omega} Z_k$.

Claim. *For every $x \in Y$ and every infinite subset $N \subset \omega$, there exists an infinite subset N' of N and a neighborhood O of x in X such that $O \cap \bigcup_{n\in N'} A_n = \emptyset$.*

Indeed, for a point $x \in Y$ choose $k \in N$ such that $x \in Z_k$. If $Z_k \cap A_n = \emptyset$ for infinitely many $n \in \omega$, it suffices to take O to be an arbitrary open set in X satisfying $O \cap Y = Z_k$. Otherwise, since $\beta_k < \alpha$, our inductive assumption implies that there exists an infinite subset N' of N such that the set $T = Z_k \cap \bigcup_{n\in N'} A_n$ is discrete. Notice that the set T is infinite. Let P and Q be infinite disjoint subsets of N'. Since T is discrete and C^*-embedded in X, the sets $A = Z_k \cap \bigcup_{n\in P} A_n$ and $B = Z_k \cap \bigcup_{n\in Q} A_n$ have disjoint closures in X. Hence the closure in X of one of the sets $\bigcup_{n\in P} A_n$ or $\bigcup_{n\in Q} A_n$ does not contain the point x. This proves our claim.

We are going to find a faithfully indexed sequence of integers $\{n_i : i \in \omega\}$ such that $A_{n_i} \cap \overline{\bigcup_{j>i} A_{n_j}} = \emptyset$ for every $i \in \omega$. This will imply that $\bigcup_{i\in\omega} A_{n_i}$ is discrete.

Put $n_0 = 0$ and $N_0 = \omega$. The set A_{n_0} being finite, there exists $k_0 \in \omega$ such that $A_{n_0} \subset Z_{k_0}$. It follows from the above Claim that there exists an infinite subset N_1 of N_0 such that A_{n_0} is disjoint from the closure of $\bigcup_{n\in N_1} A_n$. Notice that $n_0 \notin N_1$. We take an arbitrary integer $n_1 \in N_1$. Clearly, $n_1 \neq n_0$. Take an integer $k_1 > k_0$ such that $A_{n_1} \subset Z_{k_1}$.

Assume that for some $i \geq 1$ we have defined pairwise distinct integers $n_0, \dots, n_i$, integers $k_0 < \cdots < k_i$ and infinite subsets $N_0 \supset N_1 \supset \cdots \supset N_i$ of ω with $A_{n_j} \subset Z_{k_j}$ for every $j = 0, \dots, i$, $n_j \in N_j$ for every $j = 0, 1, \dots, i$, and $A_{n_j} \cap \overline{\bigcup_{n\in N_{j+1}} A_n} = \emptyset$ for every $j = 0, \dots, i-1$. Applying the above Claim to the points of A_{n_i} we find an infinite subset N_{i+1} of N_i such that A_{n_i} is disjoint from the closure of the set $\bigcup_{n\in N_{i+1}} A_n$. Take an element $n_{i+1} \in N_{i+1}$ distinct from n_j for each $j \leq i$ and choose $k_{i+1} > k_i$ such that $A_{n_{i+1}} \subset Z_{k_{i+1}}$. This finishes our construction.

Fix $i \in \omega$; let us show that $A_{n_i} \cap \overline{\bigcup_{j>i} A_{n_j}} = \emptyset$. Fix $x \in A_{n_i}$. It follows from our construction that $A_{n_i} \subset Z_{k_i}$ and A_{n_i} is disjoint from the closure of the set $\bigcup_{n\in N_{i+1}} A_n$. Since $n_j \in N_j$ for each j, we see that $x \notin \overline{\bigcup_{j>i} A_{n_j}}$.

Case 2. $\alpha = \beta_0 + 1$ for some countable ordinal β_0. Since the sets A_n are pairwise disjoint, there can be at most one $n \in \omega$ with $y_{\beta_0} \in A_n$ (we recall that $\{y_\nu : \nu < \alpha\}$ is an enumeration of Y, so y_{β_0} is the last element of Y in this ordering of Y). Hence there exists an infinite subset N_0 of ω such that $y_{\beta_0} \notin A_n$ for each $n \in N_0$. The order type of the set $\bigcup_{n\in N_0} A_n \subset \{y_\nu : \nu < \beta_0\}$ is less than or equal to β_0, so our inductive assumption implies that there is an infinite subset N_1 of N_0 such that the set $\bigcup_{n\in N_1} A_n$ is discrete. This completes the proof of the lemma. □

Example 3.12. There exists a precompact boolean group G with the Baire property such that the dual group $H = G_p^\wedge$ endowed with the pointwise convergence topology contains an infinite countably compact subspace without isolated points. In addition, G contains countable non-closed subgroups.

Proof. Let $\beta\omega$ be the Čech–Stone compactification of the discrete space ω. According to [14], there exists a dense countably compact subspace X of $K = \beta\omega \setminus \omega$ such that every countable subspace of X is scattered. Clearly X does not contain isolated points. Since countable subsets of K are C^*-embedded in K, the same is valid for countable subsets of X.

We claim that the group $C_p(X, \mathbb{Z}(2))$ of continuous functions on X with values in the discrete two-element group $\mathbb{Z}(2)$ has the Baire property. We will deduce this from Lemma 3.10. Indeed, consider a sequence $\{(\Delta_n, f_n) : n \in \omega\}$, where Δ_n are finite, pairwise disjoint subsets of X and the functions $f_n \colon \Delta_n \to \mathbb{Z}(2)$ are arbitrary. By Lemma 3.11, there exists an infinite subset I of ω such that the set $T = \bigcup_{n \in I} \Delta_n$ is discrete. For $i = 0, 1$, let

$$A_i = \{x \in T : f_n(x) = i \text{ for some } n \in I\}.$$

Then A_0 and A_1 are disjoint subsets of T and $T = A_0 \cup A_1$. Let g be the function on T such that $g(x) = i$ if $x \in A_i$, where $i = 0, 1$. Clearly g and f_n coincide on Δ_n, for each $n \in I$. Since T is discrete, g is continuous. Further, since countable discrete subsets of K are C^*-embedded, the closures of A_0 and A_1 in K are disjoint. The compact space K has a base of clopen (that is, closed and open) sets, so we can find disjoint clopen sets U_i in K such that $A_i \subset U_i$ for $i = 0, 1$ and $K = U_0 \cup U_1$. Let f be the function on X defined by $f(x) = i$ if $x \in X \cap U_i$, where $i = 0, 1$. Thus f extends g and coincides with f_n on Δ_n, for each $n \in I$. This implies, according to Lemma 3.10, that the group $G = C_p(X, \mathbb{Z}(2))$ has the Baire property.

Clearly G is a precompact boolean topological group. The dual group $H = G_p^\wedge$ is precompact and, by the Comfort–Ross duality, the group $H_p^\wedge$ is topologically isomorphic to G under the canonical isomorphism $\alpha_G \colon G \to H_p^\wedge$.

For every $x \in X$ and $f \in C_p(X, \mathbb{Z}(2))$, let $\hat{x}(f) = f(x)$. Then $\hat{x}$ is a continuous character on G and $B = \{\hat{x} : x \in X\}$ is a subset of H homeomorphic to X (see [2, Corollary 1.9.8]). So B is an infinite countably compact subspace of H which does not contain isolated points.

Finally we present a countable non-closed subgroup of G. Take a family $\{U_n : n \in \omega\}$ of pairwise disjoint non-empty clopen subsets of X. For every $n \in \omega$, let f_n be the characteristic function of the set U_n, so $f_n \in H$. It is clear that the functions f_n converge to the identity e of H, the constant function which takes the unique value zero. Choose $x_0 \in H \setminus \langle A \rangle$ and let $A = \{f_n : n \in \omega\}$. Then the sequence $\{g_n : n \in \omega\}$ converges to x_0, where $g_n = f_n + x_0$ for each $n \in \omega$. Let L be the subgroup of G generated by the set $\{g_n : n \in \omega\}$. It is clear that x_0 is an accumulation point of L. It is also clear that $x_0 \notin L$. Indeed, otherwise there exist non-negative integers $n_1 < \cdots < n_k$ such that $x_0 = g_{n_1} + \cdots + g_{n_k}$. If the number of summands k is even, then the latter equality means that $x_0 = f_{n_1} + \cdots + f_{n_k}$, thus contradicting our choice of the element x_0. If k is odd, the above equality is equivalent to $f_{n_1} + \cdots + f_{n_k} = e$ which is again impossible in view of our choice of the functions f_n. Therefore L is a countable non-closed subgroup of G. □

Remark 3.13. Corollary 2.8 implies that the group $H = G_p^\wedge$ in Example 3.12 is not pseudocompact. By Theorem 2.6, H must contain countable non-closed subgroups. This can also be proved by the following construction. Take a countable infinite subset C of B and denote by K the subgroup of H generated by C. Then the countable group K is not closed in H. Indeed, if K were closed in H, the intersection $F = K \cap B$ would be compact as a countable, countably compact space. Since $C \subset K$, the compact set K is infinite. Clearly this contradicts Theorem 3.3 since $H \cong G_p^\wedge$ and the precompact group G has the Baire property.

Acknowledgments

The authors acknowledge the financial support of the Spanish Ministerio de Economía y Competitividad, grants MTM 2013-42486-P and MTM 2016-79422-P. The second-listed author acknowledges support of Xunta de Galicia (Grant EM2013/016). The third-listed author was also partially supported by the Consejo Nacional de Ciencia y Tecnología (CONACyT) of Mexico, grant number CB-2012-01 178103. He thanks his hosts for hospitality and support during the research visit to the University of Navarra in April–May of 2015.

The authors are grateful to the anonymous referee for his/her suggestions, which led to several improvements on the originally submitted version of this paper, including a simpler proof of Lemma 3.1.

References

[1] S. Ardanza-Trevijano, M.J. Chasco, X. Domínguez, M.G. Tkachenko, Precompact non-compact reflexive Abelian groups, Forum Math. 24 (2012) 289–302.
[2] A.V. Arhangel'skii, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Ser. Math., vol. I, Atlantis Press and World Scientific, Paris–Amsterdam, 2008.
[3] L. Außenhofer, Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups, Dissertationes Math., vol. 384, PWN, Warszawa, 1999.
[4] L. Außenhofer, S. Gabriyelyan, On reflexive group topologies on abelian groups of finite exponent, Arch. Math. 99 (6) (2012) 583–588.
[5] W. Banaszczyk, Additive Subgroups of Topological Vector Spaces, Lecture Notes in Math., vol. 1466, Springer-Verlag, Berlin, Heidelberg, New York, 1991.
[6] M. Bruguera, M. Tkachenko, Pontryagin duality in the class of precompact groups and the Baire property, J. Pure Appl. Algebra 216 (12) (2012) 2636–2647.
[7] M.J. Chasco, D. Dikranjan, E. Martín Peinador, A survey on reflexivity of abelian topological groups, Topology Appl. 159 (9) (2012) 2290–2309.
[8] M.J. Chasco, E. Martín Peinador, V. Tarieladze, On Mackey topology for groups, Studia Math. 132 (3) (1999) 257–284.
[9] W.W. Comfort, K.A. Ross, Topologies induced by groups of characters, Fund. Math. 55 (1964) 283–291.
[10] W.W. Comfort, K.A. Ross, Pseudocompactness and uniform continuity in topological groups, Pacific J. Math. 16 (3) (1966) 483–496.
[11] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[12] J. Galindo, S. Macario, Pseudocompact group topologies with no infinite compact subsets, J. Pure Appl. Algebra 215 (2011) 655–663.
[13] S. Hernández, S. Macario, Dual properties in totally bounded Abelian groups, Arch. Math. 80 (2003) 271–283.
[14] I. Juhász, J. van Mill, Countably compact spaces all countable subsets of which are scattered, Comment. Math. Univ. Carolin. 22 (4) (1981) 851–855.
[15] M.C. McPhail, S.A. Morris, Varieties of abelian topological groups and scattered spaces, Bull. Aust. Math. Soc. 78 (3) (2008) 487–495.
[16] S. Mrówka, M. Rajagopalan, T. Soundararajan, A characterization of compact scattered spaces through chain limits, in: TOPO 72 — General Topology and its Applications, in: Lecture Notes in Math., vol. 378, 1974, pp. 288–297.
[17] E. Pytkeev, Condensations onto Compact Spaces and Complete Metric Spaces, Dissertation for obtaining the degree of Doctor of the State, Steklov Institute of Mathematics, 1979 (in Russian).
[18] S.U. Raczkowski, F.J. Trigos-Arrieta, Duality of totally bounded Abelian groups, Bol. Soc. Mat. Mex. III 7 (1) (2001) 1–12.
[19] D.J.F. Robinson, A Course in the Theory of Groups, Springer-Verlag, Berlin, 1982.
[20] M.G. Tkachenko, Compactness type properties in topological groups, Czechoslovak Math. J. 38 (1988) 324–341.
[21] M.G. Tkachenko, Self-duality in the class of precompact groups, Topology Appl. 156 (12) (2009) 2158–2165.
[22] M.G. Tkachenko, Topological groups in which all countable subgroups are closed, Topology Appl. 159 (7) (2012) 1806–1814.
[23] F.J. Trigos-Arrieta, Continuity, boundedness, connectedness and the Lindelöff property for topological groups, J. Pure Appl. Algebra 70 (1991) 199–210.