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Splittings and cross-sections in topological groups

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ABSTRACT

This paper deals with the splitting of extensions of topological abelian groups. Given topological abelian groups G and H, we say that $\operatorname{Ext}(G, H)$ is trivial if every extension of topological abelian groups of the form $1 \to H \to X \to G \to 1$ splits. We prove that $\operatorname{Ext}(A(Y), K)$ is trivial for any free abelian topological group $A(Y)$ over a zero-dimensional k_ω-space Y and every compact abelian group K. Moreover we show that if K is a compact subgroup of a topological abelian group X such that the quotient group X/K is a zero-dimensional k_ω-space, then there exists a continuous cross section from X/K to X. In the second part of the article we prove that $\operatorname{Ext}(G, H)$ is trivial whenever G is a product of locally precompact abelian groups and H has the form $\mathbb{T}^\alpha \times \mathbb{R}^\beta$ for arbitrary cardinal numbers α and β. An analogous result is true if $G = \prod_{i \in I} G_i$ where each G_i is a dense subgroup of a maximally almost periodic, Čech-complete group for which both $\operatorname{Ext}(G_i, \mathbb{R})$ and $\operatorname{Ext}(G_i, \mathbb{T})$ are trivial.

1. Introduction

In this paper we consider the splitting of extensions of topological abelian groups. An extension of topological abelian groups is a short exact sequence $1 \to H \to X \to G \to 1$, where H, X, G are topological abelian groups and all maps in the sequence are assumed to be continuous and open homomorphisms when considered as maps onto their images. Throughout this paper we will refer to it simply as "an extension". The extension splits if it is equivalent to $1 \to H \to H \times G \to G \to 1$ in the natural sense; this means that H splits as a subgroup of X. The *splitting problem* can be formulated as the problem of finding conditions on H and G under which all such extensions split. If this property holds we will say that $\operatorname{Ext}(G, H)$ is trivial, where $\operatorname{Ext}(G, H)$ stands for the set of equivalence classes of extensions of the form $1 \to H \to X \to G \to 1$.

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Moskowitz [14] studied this problem in the realm of locally compact abelian groups (from now on LCA groups). He proved that if G is a LCA group and $H \cong \mathbb{T}^\alpha \times \mathbb{R}^n$, for some non-negative integer n and an arbitrary cardinal α, then $\mathrm{Ext}(G, H)$ is trivial. Later on Fulp and Griffith established that a LCA group H has the property that $\mathrm{Ext}(G, H)$ is trivial for all connected LCA groups G if and only if $H \cong \mathbb{T}^\alpha \times \mathbb{R}^n$ (Theorem 3.3 in [13]).

Some particular extensions of not necessarily locally compact topological abelian groups were studied by Cabello in [7]. He introduced the concept of a quasi-homomorphism in the category of topological groups. He found that every quasi-homomorphism $q\colon G \to H$ induces an extension which he denotes by $1 \to H \to H \oplus_q G \to G \to 1$, and that every extension of this form splits provided that H is $\mathbb{R}$ or $\mathbb{T}$ and G is a product of locally compact abelian groups. The notion of quasi-homomorphism is based on that of quasi-linear map which was studied by Domański [10] in the framework of topological vector spaces.

Topological vector spaces, when considered in their additive structure, constitute an important class of topological abelian groups for which this theory is fairly well understood, at least in some concrete cases. Namely, there are well-known necessary and sufficient conditions on the spaces E and F under which every extension of Fréchet spaces $0 \to F \to L \to E \to 0$ splits [5,17]. These results have many applications, for instance to problems concerning partial differential or convolution operators.

The paper is organized as follows. We prove in Section 2 that $\mathrm{Ext}(G, K)$ is trivial whenever K is a compact abelian group and G is the free abelian topological group over a zero-dimensional k_ω-space. As a by-product we obtain the following result, which is interesting in itself: If K is a compact subgroup of a topological abelian group X such that the quotient group X/K is a zero-dimensional k_ω-space, then there exists a continuous cross section from X/K to X.

The main result of Section 3 is Theorem 3.13 which states that $\mathrm{Ext}(G, H)$ is trivial whenever $H = \mathbb{T}^\alpha \times \mathbb{R}^\beta$ with α and β arbitrary cardinal numbers and $G = \prod_{i \in I} G_i$ where each G_i is a dense subgroup of a maximally almost periodic, Čech-complete group for which both $\mathrm{Ext}(G_i, \mathbb{R})$ and $\mathrm{Ext}(G_i, \mathbb{T})$ are trivial. An important ingredient in the proof of this result is Theorem 3.5, which establishes that $\mathrm{Ext}(G, M)$ is trivial whenever G is any topological abelian group, M is metrizable and locally compact, and $\mathrm{Ext}(G/P, M)$ is trivial for each P in a cofinal family of admissible subgroups of G.

1.1. Notation, terminology, and preliminary facts

As usual, ω is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers. The unit circle of $\mathbb{C}$ with the topology inherited from $\mathbb{C}$ is denoted by $\mathbb{T}$.

We are mainly interested in abelian groups, although some of our results are valid in a more general setting. If H is a closed subgroup of a topological group G, then G/H is the space of left cosets of H with the quotient topology. This is of course a topological group when H is a normal subgroup of G.

We use multiplicative notation for the group operation. Accordingly, we denote the neutral element of a group G by 1_G or simply by 1 if there is no risk of confusion. Given a topological group G, we will denote by $\mathcal{N}_1(G)$ the family of all neighborhoods of 1 in G.

A topological abelian group G is said to be a MAP (maximally almost periodic) group if the continuous homomorphisms of G to $\mathbb{T}$ separate points of G.

A topological space X is a k_ω-space if it has the weak topology with respect to an increasing sequence of compact subsets whose union is X.

By the *character* (resp. *pseudocharacter*) of a point x in a topological space X we mean the minimum cardinality of a basis of neighborhoods of x in X (resp. of a family of open neighborhoods of x whose intersection is $\{x\}$). Every point of a topological group G has the same (pseudo)character and we refer to it simply as the *(pseudo)character of* G.

A topological space X is said to be *almost metrizable* if every $x \in X$ is contained in a compact subset which has countable character in X. A topological group G is almost metrizable if and only if it contains a compact subgroup K such that the quotient space G/K is metrizable. The classes of Čech-complete topological groups and of almost metrizable, Raĭkov complete groups coincide (for more details see [2, Section 4.3], where almost metrizable groups are called *feathered*).

The Raĭkov completion of a Hausdorff topological group G is denoted by ϱG. For any continuous homomorphism $f: G \to H$ of Hausdorff topological groups, there is a unique continuous homomorphism $\varrho f: \varrho G \to \varrho H$ extending f; if in addition f is a topological isomorphism then so is ϱf.

Following [16, 2.19] we will say that a subgroup N of a topological group G is *admissible* if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the neutral element 1 in G such that $U_{n+1}^3 \subseteq U_n$, for each $n \in \omega$, and $N = \bigcap_{n\in\omega} U_n$. It is easy to see that every admissible subgroup is closed and that every neighborhood of 1_G contains an admissible subgroup of G.

Recall that for a completely regular Hausdorff space X, the free abelian topological group over X is the free abelian group $A(X)$ endowed with the unique Hausdorff group topology for which the mapping $\eta : X \to A(X)$, which maps the topological space X onto a basis of $A(X)$, becomes a topological embedding and such that for every continuous mapping $f : X \to G$, where G is an Abelian Hausdorff group, the unique group homomorphism $\widetilde{f} : A(X) \to G$ which satisfies $f = \widetilde{f} \circ \eta$, is continuous.

A short exact sequence $E : 1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ of topological abelian groups will be called an *extension of topological groups* if both $\imath$ and π are continuous and open homomorphisms when considered as maps onto their images.

The following lemma is a standard result; the details can be found for instance in [12, Sect. 50] (for the algebraic part) or [8, Lemma 3.2].

Lemma 1.1. *Let $1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ be an extension of topological abelian groups, Y a topological abelian group, and $t: H \to Y$ a continuous homomorphism. There exists a diagram of the form*

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "\imath"] \arrow[d, "t"] & X \arrow[r, "\pi"] \arrow[d, "s"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & Y \arrow[r, "r"] & PO \arrow[r, "p"] & G \arrow[r] & 1
\end{tikzcd}$$

where PO, r and s form the push-out of $\imath$ and t in the category of topological abelian groups, and the bottom sequence is an extension of topological groups.

We say that an extension $E : 1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ *splits* if there exists a continuous homomorphism $T: X \to H \times G$ making the following diagram commutative (here $\imath_H$ and π_G are the canonical mappings).

$$\begin{tikzcd}
 & & X \arrow[dr, "\pi"] \arrow[dd, "T"] & & \\
1 \arrow[r] & H \arrow[ur, "\imath"] \arrow[dr, "\imath_H"'] & & G \arrow[r] & 1 \\
 & & H \times G \arrow[ur, "\pi_G"'] & &
\end{tikzcd}$$

It is known that if such a T exists, it must actually be a topological isomorphism. In other words, E splits if and only if it is equivalent to the trivial extension $E_0 : 1 \to H \xrightarrow{\imath_H} H \times G \xrightarrow{\pi_G} G \to 1$. Note that the extension E splits if and only if $\imath(H)$ splits as a subgroup of X.

If G and H are topological abelian groups, we will express the fact that every extension of the form $1 \to H \to X \to G \to 1$ splits by saying that $\mathrm{Ext}(G, H)$ is trivial. This notation is of course borrowed from the algebraic counterpart of this theory.

The following characterization is essential when dealing with extensions of topological abelian groups (see [8, Lemma 3.1]):

Theorem 1.2. *Let $E: 1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ be an extension of topological abelian groups. The following are equivalent:*

(1) *E splits.*
(2) *There exists a right inverse for π, i.e. a continuous homomorphism $S\colon G \to X$ with $\pi \circ S = \mathrm{Id}_G$.*
(3) *There exists a left inverse for $\imath$, i.e. a continuous homomorphism $P\colon X \to H$ with $P \circ \imath = \mathrm{Id}_H$.*

If G and H are topological abelian groups we say that a map $q\colon G \to H$ is a *quasi-homomorphism* if $q(1_G) = 1_H$ and the corresponding map $\Delta_q\colon G \times G \to H$ defined by $\Delta_q(x, y) = q(xy)q(x)^{-1}q(y)^{-1}$ is continuous at the identity of $G \times G$. A quasi-homomorphism $q\colon G \to H$ is *approximable* if there exists a homomorphism $h\colon G \to H$ such that the map $x \mapsto q(x)h(x)^{-1}$ is continuous at 1_G.

This is a generalization of the concept of (approximable) quasi-linear maps between topological linear spaces [10]. The notion of a quasi-homomorphism was first considered in the present form by Cabello in [7], where he established the following connection between quasi-homomorphisms and extensions:

Proposition 1.3. *(See [7, Lemmas 2, 3, 10].) Let G and H be topological abelian groups.*

(1) *If $q\colon G \to H$ is a quasi-homomorphism, then the sets*

$$W(V, U) = \{(h, g) \in H \times G : g \in U,\ h \in q(g) \cdot V\},$$

with $U \in \mathcal{N}_1(G)$ and $V \in \mathcal{N}_1(H)$, form a basis of neighborhoods at the identity element for a topological group topology τ_q on $H \times G$.
If $H \oplus_q G$ denotes the group $H \times G$ endowed with the topology τ_q and $\imath_H$ and π_G are the canonical inclusion and projection, respectively, then $1 \to H \xrightarrow{\imath_H} H \oplus_q G \xrightarrow{\pi_G} G \to 1$ is an extension of topological abelian groups.
(2) *An extension of topological abelian groups $1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ is equivalent to one induced by a quasi-homomorphism in the sense of (1) if and only if it splits algebraically and there exists a map $\rho\colon G \to X$ such that $\pi \circ \rho = \mathrm{Id}_G$, $\rho(1) = 1$, and ρ is continuous at the identity.*
(3) *A quasi-homomorphism $q\colon G \to H$ is approximable if and only if the corresponding induced topological extension defined in (1) splits.*

Proposition 1.4. *Let G be a metrizable topological abelian group and M a metrizable, divisible topological abelian group.*

(1) *Every extension $1 \to M \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ is equivalent to one induced by a quasi-homomorphism in the sense of Proposition 1.3.*
(2) $\mathrm{Ext}(G, M)$ *is trivial if and only if every quasi-homomorphism $q\colon G \to M$ is approximable.*

Proof. The first part is Corollary 32(1) in [4]. The second part is a consequence of the first one and Proposition 1.3(3). □

Lemma 1.5. *Let G and M be topological abelian groups. Let also H be a closed subgroup of G such that every continuous homomorphism of H to M extends to a continuous homomorphism of G to M. If $\operatorname{Ext}(G, M)$ is trivial then $\operatorname{Ext}(G/H, M)$ is trivial as well.*

Proof. This result for $M = \mathbb{T}$ is Theorem 21(2) in [4]. The same proof, with the obvious replacements, works in the general case. □

Proposition 1.6. *Let G be a locally compact abelian group and let M be either $\mathbb{R}$ or $\mathbb{T}$. Then $\operatorname{Ext}(G, M)$ is trivial.*

Proof. Observe that local compactness is a three-space property and that both $\mathbb{R}$ and $\mathbb{T}$ are universally splitting in the class of locally compact abelian groups [3, Theorem 6.16]. □

For the following result see Lemmas 4 and 6 in [7].

Lemma 1.7. *Let G be a topological abelian group and $M = \mathbb{R}$ (resp., $M = \mathbb{T}$). Let $W = [-1/3, 1/3]$ (resp., $W = \{\exp(\pi i x) : x \in [-1/6, 1/6]\}$). If $q : G \to M$ is a map such that $q(xy)q(x)^{-1}q(y)^{-1} \in W$ for every $x, y \in G$, then there exists a homomorphism $h\colon G \to M$ such that $q(x)h(x)^{-1} \in W$ for every $x \in G$.*

Proposition 1.8. *Let M be either $\mathbb{R}$ or $\mathbb{T}$. Let $(G_i)_{i\in I}$ be a family of topological abelian groups such that for every $i \in I$, every quasi-homomorphism of G_i to M is approximable. Then every quasi-homomorphism of $\prod_{i\in I} G_i$ to M is approximable.*

Proof. We start with the case of a two-element index set I. Let G and H be topological abelian groups with the property specified in the proposition and fix a quasi-homomorphism $q\colon G \times H \to M$. This means that the map $\Delta_q\colon G \times H \times G \times H \to M$ defined by $\Delta_q(g, h, g', h') = q(gg', hh')q(g, h)^{-1}q(g', h')^{-1}$ is continuous at the identity. Making use of $\Delta_q(\cdot, 1_H, \cdot, 1_H)$ we deduce that $q(\cdot, 1_H)$ is a quasi-homomorphism. Similarly, $q(1_G, \cdot)$ is a quasi-homomorphism. By hypothesis, there exist continuous homomorphisms $f_1\colon G \to M$ and $f_2\colon H \to M$ such that both $q(\cdot, 1_H)f_1(\cdot)^{-1}$ and $q(1_G, \cdot)f_2(\cdot)^{-1}$ are continuous at the identity. Consider the homomorphism of $G \times H$ to M defined by $(g, h) \mapsto f_1(g)f_2(h)$. We have

$$q(g,h)f_1(g)^{-1}f_2(h)^{-1} = \left(q(g,h)q(g,1_H)^{-1}q(1_G,h)^{-1}\right) \cdot \left(q(g,1_H)f_1(g)^{-1}\right) \cdot \left(q(1_G,h)f_2(h)^{-1}\right)$$

and then we use $\Delta_q(\cdot, 1_H, 1_G, \cdot)$ to obtain that the first factor is also jointly continuous at the identity.

The proof for a finite index set I is an easy induction, while the proof for an arbitrary set I is similar to the last step in the proof of [7, Theorem 1].

Indeed, let $q\colon \prod_{i\in I} G_i \to M$ be a quasi-homomorphism and let W_0 be the neighborhood $\{z \in \mathbb{T} : Re(z) \geq 0\}$ (in the case $M = \mathbb{T}$) or $[-1, 1]$ (in the case $M = \mathbb{R}$). Let W be as in Lemma 1.7. Find a finite subset $J_0 \subseteq I$ and neighborhoods U_i, for each $i \in J_0$, of the identity element such that $q(xy)q(x)^{-1}q(y)^{-1} \in W$ whenever $x, y \in \prod_{i\in J_0} U_i \times \prod_{i\in I\setminus J_0} G_i$. Put $J = I \setminus J_0$ and write $\prod_{i\in I} G_i$ as $G_1 \times G_2$, where $G_1 = \prod_{i\in J_0} G_i$ and $G_2 = \prod_{i\in J} G_i$. By the previous step, there exists a homomorphism $f_1\colon G_1 \to M$ and $V_i \in \mathcal{N}_1(G_i)$ for each $i \in J_0$ such that $q(g, 1)f_1(g)^{-1} \in W$ whenever $g \in \prod_{i\in J_0} V_i$. We may assume that $V_i \subseteq U_i$. Further, Lemma 1.7 implies that there is a continuous homomorphism $f_2\colon G_2 \to M$ such that the image of $q(1, \cdot)f_2(\cdot)^{-1}$ is contained in W. There exist neighborhoods O_1 and O_2 of the identities in G_1 and G_2, resp., such that for every $g \in O_1$ and $h \in O_2$, we can express $q(g, h)f_1(g)^{-1}f_2(h)^{-1}$ as the product of three elements in W, and so $q(g, h)f_1(g)^{-1}f_2(h)^{-1}$ belongs to W_0. It only remains to apply [7, Lemma 5] for $M = \mathbb{R}$ (or [4, Lemma 36] for $M = \mathbb{T}$) to deduce that the quasi-homomorphism $(g, h) \mapsto q(g, h)f_1(g)^{-1}f_2(h)^{-1}$ is continuous at the identity. This completes the proof. □

2. Free topological groups and splittings

In this section we show that $\mathrm{Ext}(A(Y), K)$ is trivial for every compact abelian group K and every free abelian topological group $A(Y)$ on a zero-dimensional k_ω-space Y.

Definition 2.1. Let $p\colon G \to H$ be a continuous homomorphism of topological groups. We say that p is a *projection along a metrizable factor* if G admits an isomorphic topological embedding j into the product $H \times M$, where M is a metrizable topological group, such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr, "p"'] & H \times M \arrow[d, "p_H"] \\ & H \end{tikzcd}$$

Here p_H stands for the projection of $H \times M$ onto the first factor.

Lemma 2.2. *If N is an admissible subgroup of a topological (not necessarily abelian) group X, then the quotient space X/N has countable pseudocharacter.*

Proof. There exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity element 1 in X such that $U_{n+1}^3 \subseteq U_n$ for each $n \in \omega$ and $N = \bigcap_{n\in\omega} U_n$. Let $\pi\colon X \to X/N$ be the quotient map onto the left coset space X/N. Then $\pi^{-1}\pi(U_{n+1}) = U_{n+1}N \subseteq U_{n+1}^2 \subseteq U_n$, for each $n \in \omega$. Therefore,

$$\pi^{-1}\left(\bigcap_{n\in\omega} \pi(U_{n+1})\right) \subseteq \bigcap_{n\in\omega} U_n = N,$$

i.e. the set $\bigcap_{n\in\omega} \pi(U_{n+1})$ contains only the element $\pi(1)$. Since the space X/N is homogeneous, we conclude that it has countable pseudocharacter. $\square$

The next two results apply in the proof of Lemma 2.5. The first of them guarantees the existence of a coarser metrizable topological group topology on certain topological abelian groups, while the second is a consequence of Michael's Selection Theorem.

Lemma 2.3. *(See [2, Corollary 3.4.26].) Suppose that G is a topological abelian group of countable pseudocharacter. Then G admits a coarser metrizable topological group topology.*

Lemma 2.4. *(See [2, Lemma 4.1.4(b)].) Let Z be a topological space, M a metrizable compact space, X a closed subspace of $Z \times M$, and $p\colon X \to Z$ the restriction to X of the projection of $Z \times M$ to the first factor. Assume that p is open and onto. Further, let f be a continuous map of a zero-dimensional compact Hausdorff space Y onto Z, A a closed subset of Y and h a continuous map of A to X such that $p \circ h = f\restriction_A$. Then there exists a continuous map $\overline{h}\colon Y \to X$ with $\overline{h}\restriction_A = h$ and such that $p \circ \overline{h} = f$.*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, dashed, "\overline{h}"] & Z \\ A \arrow[u, hook] \arrow[r, "h"'] & X \arrow[u, "p"'] \end{tikzcd}$$

The second part of the following lemma will be generalized in Theorem 2.8 where we drop the metrizability restriction on the group K.

Lemma 2.5. *Let K be a compact metrizable subgroup of a topological abelian group X and $p\colon X \to X/K$ the quotient homomorphism. Then p is a projection along a metrizable factor. Furthermore, if a zero-dimensional subspace Y of X/K is a k_ω-space, then there exists a continuous map $s\colon Y \to X$ satisfying $p \circ s = \mathrm{Id}_Y$.*

Proof. Our argument imitates the one in [2, Section 4.1]. Since the group K is metrizable, it has a countable local base at the identity element, say, $\{V_n : n \in \omega\}$. Then there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the neutral element 1 in X such that $U_{n+1}^3 \subseteq U_n$ and $U_n \cap K \subseteq V_n$, for each $n \in \omega$. Clearly $N = \bigcap_{n\in\omega} U_n$ is an admissible subgroup of X and $N \cap K = \bigcap_{n\in\omega} U_n \cap K \subseteq \bigcap_{n\in\omega} V_n = \{1\}$. By Lemma 2.2, the quotient group X/N has countable pseudocharacter. Therefore, by Lemma 2.3, it admits a coarser metrizable topological group topology $\mathcal{T}$. We denote the topological group $(X/N, \mathcal{T})$ by M. Clearly, the coset map $\pi\colon X \to M$, $\pi(x) = xN$, is continuous.

Since K is compact, the quotient map $p\colon X \to X/K$ is perfect. Put $G = X/K$ and denote by j the diagonal product of the homomorphisms p and π. Then j is a perfect homomorphism of X to the product group $G \times M$ since the diagonal product of a perfect map and a continuous map is perfect [11, Theorem 3.7.11]. It is easy to see that j is injective. Indeed, take an arbitrary element $x \in X$ distinct from 1. If $x \notin K$, then $p(x) \neq 1$ and hence $j(x) \neq 1$. If $x \in K$, then $x \notin N$, whence it follows that $\pi(x) \neq 1$ and $j(x) \neq 1$. Thus j is a perfect one-to-one homomorphism of X onto $j(X)$, and so j is a topological isomorphism of X onto the subgroup $j(X)$ of $G \times M$. We denote by p_G and p_M the projections of $G \times M$ onto G and M, respectively. The following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, "j"] & G \\
M & G \times M \arrow[l, "p_M"] \arrow[u, "p_G"']
\end{tikzcd}$$

Since $p = p_G \circ j$, we see that p is a projection along the metrizable factor M.

As Y is a k_ω-space, we can represent it as the direct limit of an increasing sequence $\{Y_n : n \in \omega\}$ of compact subspaces. We construct the required map $s\colon Y \to X$ by induction. Clearly Y_0 is a compact zero-dimensional subspace of Y. Since p is perfect, $X_0 = p^{-1}(Y_0)$ is a compact subspace of X and $K_0 = p_M(j(X_0))$ is a compact subspace of M. Thus $j(X_0)$ is a compact subspace of $Y_0 \times K_0$, where K_0 is a compact metrizable space.

The restriction of p to X_0 is a continuous open map of X_0 onto Y_0 and, therefore, the restriction of p_G to $j(X_0)$ is a continuous open map of $j(X_0)$ onto Y_0. We obtain the following commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "p\restriction_{X_0}"] \arrow[dr, "j\restriction_{X_0}"'] & Y_0 \\
& j(X_0) \arrow[u, "p_G\restriction_{j(X_0)}"']
\end{tikzcd} \tag{1}$$

By Lemma 2.4 (with $A = \emptyset$, $Y = Z = Y_0$, $f = \mathrm{Id}_{Y_0}$ and $p = p_G\restriction_{j(X_0)}$), there exists a continuous map $t_0\colon Y_1 \to j(X_0)$ such that $p_G \circ t_0 = \mathrm{Id}_{Y_0}$. Then by the commutativity of (1), $s_0 = j^{-1} \circ t_0$ is a continuous map of Y_0 to X_0 satisfying $p \circ s_0 = \mathrm{Id}_{Y_0}$.

Suppose that for some $n \in \omega$, we have defined a continuous map $s_n\colon Y_n \to X_n = p^{-1}(Y_n)$ satisfying $p \circ s_n = \mathrm{Id}_{Y_n}$. Then $t_n = j \circ s_n$ is a continuous map of Y_n to $j(X_n)$. The map t_n satisfies $p_G \circ t_n =$

$p{\restriction}_{X_n} \circ s_n = \mathrm{Id}_{Y_n}$. Note that $X_{n+1} = p^{-1}(Y_{n+1})$ and $j(X_{n+1})$ are compact subspaces of X and $G \times M$, respectively. Hence $K_{n+1} = p_M(j(X_{n+1}))$ is a compact subspace of M and $j(X_{n+1}) \subseteq Y_{n+1} \times K_{n+1}$. We obtain the following commutative diagram

$$\begin{tikzcd} Y_{n+1} \arrow[r, "\mathrm{Id}_{Y_{n+1}}"] & Y_{n+1} \\ Y_n \arrow[u, hook] \arrow[r, "t_n"] & j(X_{n+1}) \arrow[u, "p_G{\restriction}_{j(X_{n+1})}"'] \end{tikzcd}$$

By Lemma 2.4 (with $A = Y_n$ and $h = t_n$), there exists a continuous map t_{n+1} of Y_{n+1} to $j(X_{n+1})$ which extends t_n and satisfies $p_G \circ t_{n+1} = \mathrm{Id}_{Y_{n+1}}$. Then the continuous map $s_{n+1} = j^{-1} \circ t_{n+1}$ of Y_{n+1} to X_{n+1} extends s_n and satisfies $p \circ s_{n+1} = \mathrm{Id}_{Y_{n+1}}$. This finishes our construction.

Let s be the map of Y to X which coincides with s_n on Y_n for each $n \in \omega$. Our choice of the representation $Y = \bigcup_{n\in\omega} Y_n$ implies that s is continuous. It is also clear from the construction that $p \circ s = \mathrm{Id}_Y$, and the proof is complete. $\square$

In the rest of the section we use the language of spectral representations and inverse systems. Let us recall the definition of spectral representations.

Let X be a space, κ a cardinal number, and let p_α be a quotient map of X onto a space X_α, for each $\alpha < \kappa$, such that the following two conditions are satisfied:

(S1) If $x, y \in X$, $\alpha, \beta \in \kappa$ and $\alpha < \beta$, then $p_\beta(x) = p_\beta(y)$ implies $p_\alpha(x) = p_\alpha(y)$;
(S2) If $x, y \in X$ are distinct, then $p_\alpha(x) \neq p_\alpha(y)$, for some $\alpha \in \kappa$.

Then we will say that $\mathcal{P} = \{p_\alpha : \alpha < \kappa\}$ is a *spectral representation* of the space X.

The next proposition that will be used in the proof of Theorem 2.8 is a well-known result, which appears in [1]. We present its proof here because [1] is not available in English.

Proposition 2.6. *(See [1, Addendum to Chapter 1].) Let κ be a cardinal number and $\{p_\alpha : \alpha < \kappa\}$ a spectral representation of a space X. For $\alpha, \beta < \kappa$ with $\alpha < \beta$, let $p_{\beta,\alpha} = p_\alpha \circ p_\beta^{-1} : p_\beta(X) \to p_\alpha(X)$. If the projection p_0 is a perfect map, then X is naturally homeomorphic to the limit space of the inverse system $\{X_\alpha, p_{\beta,\alpha} : \alpha < \beta < \kappa\}$, where $X_\alpha = p_\alpha(X)$ for each $\alpha < \kappa$.*

Proof. Denote by p the diagonal product of the family $\{p_\alpha : \alpha < \kappa\}$. Since p_0 is perfect, the map p is perfect by [11, Theorem 3.7.11]. Hence the image $Y = p(X)$ is a closed subspace of the product $\Pi = \prod_{\alpha<\kappa} X_\alpha$, where $X_\alpha = p_\alpha(X)$ for each $\alpha < \kappa$. It also follows from condition (S2) that p is one-to-one. As p is perfect and one-to-one, it is a homeomorphism of X onto Y. For every $\alpha < \kappa$, let $\pi_\alpha : \Pi \to X_\alpha$ be the projection. The limit space of the inverse system $\{X_\alpha, p_{\beta,\alpha} : \alpha < \beta < \kappa\}$, say, L is the subspace of Π which consists of all $x \in \Pi$ such that $p_{\beta,\alpha}(\pi_\beta(x)) = p_\alpha(x)$ whenever $\alpha < \beta < \kappa$. It is clear that $Y \subseteq L$. According to [9, Corollary 1.2.5], Y is a dense subspace of L. Since Y is closed in Π, we see that $Y = L$, i.e. $p : X \to L$ is a homeomorphism. $\square$

Definition 2.7. Let X be a topological group, H a closed subgroup of X and $p : X \to X/H$ be the quotient map. A *continuous cross section* from X/H to X is a continuous map $s : X/H \to X$ such that $p \circ s = \mathrm{Id}_{X/K}$.

Theorem 2.8. *Let K be a compact subgroup of a topological abelian group X and $p : X \to X/K$ the quotient homomorphism. If a zero-dimensional subspace Y of X/K is a k_ω-space, then there exists a continuous map $s : Y \to X$ satisfying $p \circ s = \mathrm{Id}_Y$. In particular, if X/K is a zero-dimensional k_ω-space, there exists a continuous cross section from X/K to X.*

Proof. Since K is compact, the homomorphism p is perfect [2, Theorem 1.5.7]. Let τ be the character of X, i.e. the minimum cardinality of a local base at the identity element of X. Choose a family $\{N_\alpha : \alpha < \tau\}$ of admissible subgroups of X such that

$(*)$ every neighborhood of the neutral element 1 in X contains N_α, for some $\alpha < \tau$.

For every $\alpha < \tau$, we denote by φ_α the quotient homomorphism of X onto X/N_α.

Put $\pi_0 = p$, and if $0 < \alpha < \tau$, denote by π_α the diagonal product of p and ϕ_α, where ϕ_α is in turn the diagonal product of the homomorphisms φ_β with $\beta < \alpha$. Then π_α is a continuous homomorphism of X to the product group $X/K \times \left(\prod_{\beta<\alpha} X/N_\beta\right)$.

It follows from [11, Theorem 3.7.11] that π_α is a perfect homomorphism of X onto the subgroup $X_\alpha = \pi_\alpha(X)$ of $X/K \times \phi_\alpha(X)$. It follows from $(*)$ that the homomorphisms φ_α with $0 < \alpha < \tau$ separate points of X. Hence the diagonal product of p and the family $\{\varphi_\alpha : \alpha < \tau\}$ is a topological isomorphism of X onto its image.

Given ordinals α, β with $\beta < \alpha < \tau$, we denote by $\pi_{\alpha,\beta}$ the canonical homomorphism of $X_\alpha = \pi_\alpha(X)$ onto $X_\beta = \pi_\beta(X)$ satisfying $\pi_\beta = \pi_{\alpha,\beta} \circ \pi_\alpha$. Then $\mathcal{P} = \{X_\alpha, \pi_{\alpha,\beta} : \beta < \alpha < \tau\}$ is an inverse system of topological abelian groups and $\{\pi_\alpha : \alpha < \tau\}$ is a spectral representation of X. Since the homomorphisms π_α are perfect, Proposition 2.6 implies that the group X is topologically isomorphic to the limit group of the inverse system $\mathcal{P}$. Similarly, X_α is topologically isomorphic to the limit group of the inverse system $\{X_\beta, \pi_{\beta,\gamma} : \gamma < \beta < \alpha\}$ for each limit ordinal α satisfying $\omega \le \alpha < \tau$.

For each $\alpha < \tau$, denote by K_α the kernel of the homomorphism π_α. Then $K_\alpha \subseteq K$ is a compact group and the kernel of the bonding homomorphism $\pi_{\alpha+1,\alpha} \colon X_{\alpha+1} \to X_\alpha$ is topologically isomorphic to the compact group $K_\alpha/K_\alpha \cap N_\alpha$. The quotient group $K_\alpha/(K_\alpha \cap N_\alpha)$ is also metrizable because it is a compact space with countable pseudocharacter. Hence the homomorphism $\pi_{\alpha+1,\alpha}$ satisfies the hypothesis of Lemma 2.5.

We are going to define a system of continuous maps $s_\alpha \colon Y \to X_\alpha$ satisfying the following condition for all α, β with $0 \le \beta < \alpha < \tau$:

$(**)$ $\pi_{\alpha,\beta} \circ s_\alpha = s_\beta$.

Let us start with letting $s_0 = \mathrm{Id}_Y$. Suppose that the system $\{s_\beta : \beta < \alpha\}$ satisfying $(**)$ is defined for some ordinal α with $0 < \alpha < \tau$. If α is limit, then since X_α is isomorphic to the limit of the inverse system $\{X_\beta, \pi_{\beta,\gamma} : \gamma < \beta < \alpha\}$, there exists a unique continuous map $s_\alpha \colon Y \to X_\alpha$ such that $\pi_{\alpha,\beta} \circ s_\alpha = s_\beta$ for each $\beta < \alpha$. Suppose now that α is a successor ordinal, say, $\alpha = \nu + 1$. It follows from $\pi_{\nu,0} \circ s_\nu = \mathrm{Id}_Y$ that s_ν is a homeomorphism of Y onto a subspace of X_ν. In particular, $Y_\nu = s_\nu(Y)$ is a zero-dimensional k_ω-subspace of X_ν. By applying Lemma 2.5 to the open homomorphism $\pi_{\nu+1,\nu} = \pi_{\alpha,\nu}$, we deduce that there exists a continuous map $t_\nu \colon Y_\nu \to X_\alpha$ such that $\pi_{\alpha,\nu} \circ t_\nu = \mathrm{Id}_{Y_\nu}$. Let us put $s_\alpha = t_\nu \circ s_\nu$. It is clear that the system $\{s_\beta : 0 \le \beta \le \alpha\}$ satisfies $(**)$. This finishes our recursive construction.

Since X is the inverse limit of the system $\mathcal{P}$, it follows from $(**)$ that there exists a unique continuous map $s \colon Y \to X$ such that $\pi_\alpha \circ s = s_\alpha$ for each $\alpha < \tau$. In particular, $p \circ s = \mathrm{Id}_Y$. This completes the proof of the theorem. $\square$

The main result of this section is the following theorem:

Theorem 2.9. *Let K be a compact abelian group and $A(Y)$ the free abelian topological group on a zero-dimensional k_ω-space Y. Then $\mathrm{Ext}(A(Y), K)$ is trivial and every quasi-homomorphism $\omega \colon A(Y) \to K$ is approximable.*

Proof. Let $E : 1 \to K \xrightarrow{\imath} X \xrightarrow{p} A(Y) \to 1$ be an extension of topological groups. By Theorem 2.8, there exists a continuous map $s \colon Y \to X$ satisfying $p \circ s = \mathrm{Id}_Y$. Since $A(Y)$ is the free abelian topological group over Y, the map s extends to a continuous homomorphism $S\colon A(Y) \to X$ and an easy verification shows that $p \circ S = \mathrm{Id}_{A(Y)}$. Hence Theorem 1.2 implies that E splits, so every quasi-homomorphism of $A(Y)$ to K is approximable by Proposition 1.3. □

Theorem 2.8 also implies the following result along the lines of Proposition 1.4:

Proposition 2.10. *Let K be a divisible compact abelian group and G a topological abelian group which is a zero-dimensional k_ω-space. Every extension of topological abelian groups $1 \to K \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ is equivalent to one induced by a quasi-homomorphism. Consequently $\mathrm{Ext}(G, K)$ is trivial if and only if every quasi-homomorphism $q\colon G \to K$ is approximable.*

Proof. Fix such an extension $E : 1 \to K \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$. Since K is divisible, E splits algebraically. By Theorem 2.8, there exists a continuous map $s\colon G \to X$ satisfying $\pi \circ s = \mathrm{Id}_G$. But this continuous map can be chosen such that $s(1) = 1$, therefore by Proposition 1.3 (2) we obtain that E is equivalent to an extension of topological groups induced by a quasi-homomorphism. The second assertion follows from the first one and Proposition 1.3 (3). □

3. Products and splittings

Our aim in this section is to prove Theorem 3.13 which states the following: *Let $G = \prod_{i\in I} G_i$ be the product of a family of topological abelian groups, where each factor G_i is a dense subgroup of a MAP and Čech-complete group. If both $\mathrm{Ext}(G_i, \mathbb{R})$ and $\mathrm{Ext}(G_i, \mathbb{T})$ are trivial for each $i \in I$, then $\mathrm{Ext}(G, H)$ is trivial, where H is an arbitrary product of copies of $\mathbb{R}$ and $\mathbb{T}$.*

This theorem is the last one of this section and it can be regarded as the final step in a series of successively more general results. We start by proving (Lemma 3.1) that $\mathrm{Ext}(G, H)$ is trivial whenever H is either $\mathbb{R}$ or $\mathbb{T}$ and G is a countable product of metrizable groups G_n such that $\mathrm{Ext}(G_n, H)$ is trivial for every n.

A second, more technical step is fulfilled in Theorem 3.5 where we give a sufficient condition for $\mathrm{Ext}(G, M)$ to be trivial, where G is any topological abelian group and M is metrizable and locally compact.

The next important step is achieved by applying the sufficient condition just obtained to a product $G = \prod_{i\in I} G_i$ of almost metrizable groups G_i (Theorem 3.8), thus showing that

(1) If G_i is a MAP group and $\mathrm{Ext}(G_i, \mathbb{T})$ is trivial for every $i \in I$, then $\mathrm{Ext}(G, \mathbb{T})$ is trivial.
(2) If $\mathrm{Ext}(G_i, \mathbb{R})$ is trivial for every $i \in I$, then $\mathrm{Ext}(G, \mathbb{R})$ is trivial.

This concludes the most intricate part of the argument. In the last part of this section we present several results which can be obtained in a standard way and allow us to complete the proof of Theorem 3.13.

Lemma 3.1. *Let H be either $\mathbb{R}$ or $\mathbb{T}$. If $(G_n)_{n\in\omega}$ is a sequence of metrizable abelian groups such that $\mathrm{Ext}(G_n, H)$ is trivial for every $n \in \omega$, then $\mathrm{Ext}(\prod_{n\in\omega} G_n, H)$ is trivial.*

Proof. Clearly the group $G = \prod_{n\in\omega} G_n$ is metrizable. By Proposition 1.4(2), it suffices to prove that every quasi-homomorphism $\omega\colon G \to H$ is approximable. Taking into account Proposition 1.3(3), this follows from Proposition 1.8. □

The following important result was proved by Arhangel'skii [2, Theorem 3.2.2].

Proposition 3.2. *Suppose that G is a topological (not necessarily abelian) group, H is a locally compact subgroup of G and $\pi\colon G \to G/H$ is the natural quotient map. There exists an open neighborhood U of the* *identity element 1_G in G such that $\pi(\overline{U})$ is closed in G/H and the restriction of π to $\overline{U}$ is a perfect map of $\overline{U}$ onto the subspace $\pi(\overline{U})$ of G/H.*

The two lemmas that follow prepare the ground for the proof of Theorem 3.5.

Lemma 3.3. *Let $p\colon X \to G$ be a quotient homomorphism of topological abelian groups, where $\ker p$ is a locally compact subgroup of X. There exists an admissible subgroup N_0 of X such that for every closed subgroup N of X contained in N_0, the image $M = p(N)$ is closed in G and there exists a quotient homomorphism $\varphi_N\colon X/N \to G/M$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\pi_N"'] & G \arrow[d, "\pi_M"] \\ X/N \arrow[r, "\varphi_N"] & G/M \end{tikzcd}$$

commutes and $\ker \varphi_N = \pi_N(\ker p)$, where $\pi_N\colon X \to X/N$ and $\pi_M\colon G \to G/M$ are the quotient homomorphisms.

Proof. Since $K = \ker p$ is a locally compact subgroup of X, Proposition 3.2 implies that there exists a closed neighborhood W of the identity 1_X in X such that $p\restriction_W$ is a perfect map. We choose an admissible subgroup N_0 of X with $N_0 \subseteq W$.

Let N be a closed subgroup of X contained in N_0. Since the map $p\restriction_W$ is closed and $N \subseteq N_0 \subseteq W$, we see that the subgroup $M = p(N)$ is closed in G. Let $f = \pi_M \circ p$. Then f is a continuous homomorphism of X to G/M and $\ker f = p^{-1}p(N) = NK$. Since $\ker \pi_N = N \subseteq NK = \ker f$, there exists a homomorphism $\varphi_N\colon X/N \to G/M$ satisfying $\varphi_N \circ \pi_N = f$. Notice that f is open as a composition of two open homomorphisms, so φ_N is continuous and open. Finally, it is clear that φ_N is onto and $\ker \varphi_N = \pi_N(NK) = \pi_N(K)$. $\square$

Lemma 3.4. *Let $p\colon X \to G$ be a continuous open homomorphism of (not necessarily abelian) topological groups. If the kernel of p is locally compact, then*

(1) *There exists an admissible subgroup N_0 of X such that $p(N)$ is an admissible subgroup of G, for each admissible subgroup N of X contained in N_0.*
(2) *Let $\mathcal{L}$ be a cofinal subfamily of the family of admissible subgroups of G ordered by inverse inclusion. For every admissible subgroup N of X, there exists an admissible $N' \subseteq N$ with $p(N') \in \mathcal{L}$.*

Proof. We can assume that $p(X) = G$, otherwise we replace G with its open subgroup $p(X)$.

(1) Since $\ker p$ is a locally compact subgroup of X, it follows from Proposition 3.2 that there exists an open neighborhood U_0 of the identity 1_X in X such that the restriction of p to $\overline{U_0}$ is a perfect map. Let $\{U_n : n \in \omega\}$ be a sequence of open symmetric neighborhoods of 1 in X such that $U_{n+1}^3 \subseteq U_n$ for each $n \in \omega$. Then $N_0 = \bigcap_{n\in\omega} U_n$ is the required admissible subgroup of X.

Indeed, let N be an arbitrary admissible subgroup of X contained in N_0. Take a sequence $\{V_n : n \in \omega\}$ of symmetric open neighborhoods of 1_X in X such that $V_{n+1}^3 \subseteq V_n$ for each $n \in \omega$ and $N = \bigcap_{n\in\omega} V_n$. The sets $O_n = V_n \cap U_n$, with $n \in \omega$, are also symmetric neighborhoods of 1_X which satisfy $O_{n+1}^3 \subseteq O_n$ and $N = N_0 \cap N = \bigcap_{n\in\omega} O_n$. Then $W_n = p(O_n)$ is an open symmetric neighborhood of the neutral element in G and $W_{n+1}^3 \subseteq W_n$, for each $n \in \omega$. It is clear that $P = \bigcap_{n\in\omega} W_n$ is an admissible subgroup of G. To finish the proof it suffices to verify that $p(N) = P$.

It follows from the choice of the sets O_n that $\overline{O_{n+1}} \subseteq O_n$, for each $n \in \omega$. In particular $N = \bigcap_{n\in\omega} \overline{O_n}$. Take any point $y \in P$. Then $p^{-1}(y) \cap O_n \neq \emptyset$ for each $n \in \omega$. As $O_0 \subseteq U_0$ and the map $p\restriction_{\overline{U_0}}$ is perfect, the set $\overline{O_0} \cap p^{-1}(y)$ is compact. It follows that $\emptyset \neq p^{-1}(y) \cap \bigcap_{n\in\omega} \overline{O_n} = p^{-1}(y) \cap N$. This proves the equality $p(N) = P$.

(2) Fix an admissible subgroup $N \subseteq X$. Take an admissible subgroup N_0 of X as in (1). Then $p(N_0 \cap N)$ is an admissible subgroup of G. Since $\mathcal{L}$ is cofinal, there exists $P \in \mathcal{L}$ such that $P \subseteq p(N_0 \cap N)$. Put $N' = N_0 \cap N \cap p^{-1}(P)$. Then $N' \subseteq N$ is an admissible subgroup of X and $p(N') = p(N_0 \cap N) \cap P = P$. $\square$

Theorem 3.5. *Let M be a metrizable, locally compact abelian group. Let also G be a topological abelian group and $\mathcal{L}$ a cofinal subfamily of the family of admissible subgroups of G, ordered by inverse inclusion. If* $\mathrm{Ext}(G/P, M)$ *is trivial for each $P \in \mathcal{L}$, then* $\mathrm{Ext}(G, M)$ *is trivial.*

Proof. Suppose that $E\colon 1 \to M \overset{\imath}{\to} X \overset{\pi}{\to} G \to 1$ is an extension of topological abelian groups. Since $\ker \pi = \imath(M)$ is locally compact, we can find an admissible subgroup N_2 of X as in Lemma 3.3. Since $\imath(M)$ is metrizable, there exists an admissible subgroup N_1 of X such that $N_1 \cap \imath(M) = \{1_X\}$ and $N_1 \subseteq N_2$. By Lemma 3.4(2), we can find an admissible subgroup N_0 of X such that $N_0 \subseteq N_1$ and $P = \pi(N_0) \in \mathcal{L}$. Let the sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of 1_X in X witness the fact that N_0 is an admissible subgroup of X. Clearly $\overline{U_{n+1}} \subseteq U_n$ for each $n \in \omega$. Since the group $\imath(M)$ is locally compact and $N_0 \cap \imath(M) \subseteq N_1 \cap \imath(M) = \{1_X\}$, the family $\{U_n \cap \imath(M) : n \in \omega\}$ forms a local base at the identity in $\imath(M)$ (see [11, 3.1.5]).

Let $p\colon X \to X/N_0$ and $f\colon G \to G/P$ be the quotient homomorphisms. As $N_0 \subseteq N_1 \subseteq N_2$, by Lemma 3.3 there exists a continuous open homomorphism φ of X/N_0 onto G/P such that $f \circ \pi = \varphi \circ p$ and $\ker \varphi = p(\imath(M))$. Consider the commutative diagram

$$\begin{tikzcd}
E : 1 \arrow[r] & M \arrow[r, "\imath"] \arrow[d, "q"] & X \arrow[r, "\pi"] \arrow[d, "p"] & G \arrow[r] \arrow[d, "f"] & 1 \\
E' : 1 \arrow[r] & p(\imath(M)) \arrow[r, "\imath'"] & X/N_1 \arrow[r, "\varphi"] & G/P \arrow[r] & 1
\end{tikzcd}$$

where $\imath'$ is the canonical inclusion and $q(x) = p(\imath(x))$ for every $x \in M$. The sequence E' is also an extension of topological abelian groups. The inclusions $q^{-1}(p(\imath(M)) \cap p(U_{n+1})) \subseteq \imath^{-1}(U_n)$, where $n \in \omega$, are easy to check. Since the sequence $\{\imath^{-1}(U_n) : n \in \omega\}$ is a local base at the identity of M, we conclude that q is one-to-one and open, hence a topological isomorphism.

By hypothesis the extension E' splits. By Theorem 1.2, there exists a continuous homomorphism $R\colon X/N_1 \to p(\imath(M))$ such that $R \circ \imath' = \mathrm{Id}_{p(\imath(M))}$. It is clear that the continuous homomorphism $S\colon X \to M$ defined by $S = q^{-1} \circ R \circ p$ satisfies $S \circ \imath = \mathrm{Id}_M$. Hence the extension E splits. $\square$

The next two lemmas will be used in the proof of Theorem 3.8.

Lemma 3.6. *Let G be an almost metrizable topological abelian group. Every admissible subgroup N of G contains an admissible, compact subgroup K such that G/K is metrizable.*

Proof. Let N be an admissible subgroup of G and $(W_n)_{n\in\omega}$ be a sequence of open, symmetric neighborhoods of the identity in G such that $W_{n+1}^3 \subseteq W_n$ for each n and $\bigcap_{n\in\omega} W_n = N$. Take a compact subgroup H of G of countable character in G; let $(U_n)_{n\in\omega}$ be a basis of open neighborhoods of H in G. Find a sequence $(V_n)_{n\in\omega}$ of open symmetric neighborhoods of the identity in G such that $V_{n+1}^3 \subseteq V_n \cap U_n$ and $V_n \subseteq W_n$ for every $n \in \omega$. Put $K = \bigcap_{n\in\omega} V_n$. It is clear that K is admissible and $K \subseteq N$. By [2, Lemma 4.3.10], K is a compact subgroup of G and $(V_n)_{n\in\omega}$ is a base for K in G. Hence the quotient group G/K is metrizable. $\square$

Lemma 3.7. *Let $(G_i)_{i\in I}$ be a family of almost metrizable abelian groups and $G = \prod_{i\in I} G_i$. Let $\mathcal{L}$ be the family of subgroups of G that have the form $\prod_{i\in I} N_i$ where*

(1) *N_i is either a compact, admissible subgroup of G_i or the whole G_i*
(2) *the quotients G_i/N_i are metrizable for every i*
(3) *$N_i \neq G_i$ for at most countably many $i \in I$.*

Then $\mathcal{L}$ is a cofinal family of admissible subgroups of G.

Proof. It is clear that every $N \in \mathcal{L}$ is admissible. Conversely, let N be an admissible subgroup of G. It is easy to see that for each $i \in I$ we can find a subgroup N_i of G_i such that either $N_i = G_i$ or N_i is an admissible subgroup of G_i, the product $\prod_{i\in I} N_i$ is contained in N, and $N_i = G_i$ for all but countably many $i \in I$. Now put $N_i' = G_i$ if $N_i = G_i$, and for those i with $N_i \neq G_i$ find (using Lemma 3.6) a subgroup $N_i' \subseteq N_i$ such that N_i' is admissible, compact and G_i/N_i' is metrizable. Then $N \supseteq \prod_{i\in I} N_i' \in \mathcal{L}$. □

The following result is the key part of the proof of Theorem 3.13.

Theorem 3.8. *Let $(G_i)_{i\in I}$ be a family of almost metrizable abelian groups and $G = \prod_{i\in I} G_i$.*

(1) *If G_i is a MAP group and $\operatorname{Ext}(G_i, \mathbb{T})$ is trivial for every $i \in I$, then $\operatorname{Ext}(G, \mathbb{T})$ is trivial.*
(2) *If $\operatorname{Ext}(G_i, \mathbb{R})$ is trivial for every $i \in I$, then $\operatorname{Ext}(G, \mathbb{R})$ is trivial.*

Proof. Let M be either $\mathbb{T}$ or $\mathbb{R}$. Consider the family $\mathcal{L}$ defined in Lemma 3.7. By Theorem 3.5, it suffices to prove that for each $N \in \mathcal{L}$, every extension $1 \to M \to Y \to G/N \to 1$ splits. If $N \in \mathcal{L}$, then $N = \prod_{i\in I} N_i$, where N_i is either a compact, admissible subgroup of G_i or the whole G_i, the quotients G_i/N_i are metrizable for every i, and $N_i \neq G_i$ for at most countably many $i \in I$. Clearly $G/N \cong \prod_{i\in I} G_i/N_i$. Note that because N_i is compact every continuous homomorphism from N_i to M can be continuously extended to G_i (indeed, the case $M = \mathbb{R}$ is immediate and in the case $M = \mathbb{T}$ it is [6, Corollary 4]). Hence by Lemma 1.5, $\operatorname{Ext}(G_i/N_i, M)$ is trivial for every $i \in I$. Therefore G/N is topologically isomorphic to a countable product of metrizable topological groups G_i/N_i such that $\operatorname{Ext}(G_i/N_i, M)$ is trivial. By Lemma 3.1, $\operatorname{Ext}(G/N, M)$ is trivial. □

Next we prove a few results concerning completions of extensions and extensions by products of topological abelian groups.

Proposition 3.9. *Let G be a topological abelian group and $H \subseteq G$ a closed subgroup of G. If the Raĭkov completion of H is Čech-complete, then the canonical map $\phi\colon G/H \to \varrho G/\varrho H$ is a dense embedding which extends to a topological isomorphism of $\varrho(G/H)$ onto $\varrho G/\varrho H$.*

Proof. Let $\pi_H\colon G \to G/H$ and $\pi_{\varrho H}\colon \varrho G \to \varrho G/\varrho H$ be the canonical homomorphisms. Note that $\phi(\pi_H(g)) = \pi_{\varrho H}(g)$ for every $g \in G$. It is clear that ϕ is a continuous monomorphism. That $\phi(G/H) = \pi_{\varrho H}(G)$ is dense in $\varrho G/\varrho H$ follows from the fact that G is dense in ϱG and $\pi_{\varrho H}$ is a quotient map.

Let us see that ϕ is relatively open. Fix a closed neighborhood U of 1 in G. Let us show that for every symmetric neighborhood V of 1 in G with $VV \subseteq U$, we have $\phi(\pi_H(U)) \supset \phi(G/H) \cap \pi_{\varrho H}(\overline{V})$ or, equivalently, $\pi_{\varrho H}(U) \supset \pi_{\varrho H}(G) \cap \pi_{\varrho H}(\overline{V})$ where the closure is taken in ϱG. Fix $z \in \overline{V}$ and $g \in G$ with $zg^{-1} \in \varrho H$. Fix $h \in H \cap zg^{-1}\overline{V}$. Then $u = hg$ satisfies $u \in \overline{VV} \cap G \subseteq \overline{U} \cap G = U$ and $zu^{-1} \in \varrho H$.

The group $\varrho G/\varrho H$ is complete by [15, 11.18]. Hence $\varrho\phi\colon \varrho(G/H) \to \varrho G/\varrho H$ is a topological isomorphism. □

Proposition 3.10. *Let $E : 1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ be an extension of topological abelian groups. Suppose that the Raĭkov completion of H is a Čech-complete group. Then the sequence $\varrho E : 1 \to \varrho H \xrightarrow{\varrho\imath} \varrho X \xrightarrow{\varrho\pi} \varrho G \to 1$ is an extension of topological abelian groups.*

Proof. There exists a topological isomorphism $\alpha\colon G \to X/\imath(H)$ such that $\alpha \circ \pi = \pi_{\imath(H)}$, where $\pi_{\imath(H)}$ is the natural quotient map. Note that the completion $\varrho\alpha\colon \varrho G \to \varrho(X/\imath(H))$ of α is a topological isomorphism, too.

As a subgroup of ϱX, $\varrho(\imath(H))$ coincides with $\varrho\imath(\varrho H)$. Let $\varphi\colon \varrho(X/\imath(H)) \to \varrho X/\varrho\imath(\varrho H)$ be the topological isomorphism of Proposition 3.9. It is easy to check that the diagram

$$\begin{tikzcd}
\varrho E: & 1 \arrow[r] & \varrho H \arrow[r, "\varrho\imath"] \arrow[d, "\widetilde{\varrho\imath}"] & \varrho X \arrow[r, "\varrho\pi"] \arrow[d, equal] & \varrho G \arrow[r] \arrow[d, "\varphi\circ\varrho\alpha"] & 1 \\
E': & 1 \arrow[r] & \varrho\imath(\varrho H) \arrow[r] & \varrho X \arrow[r] & \varrho X/\varrho\imath(\varrho H) \arrow[r] & 1
\end{tikzcd}$$

commutes, where E' is the canonical extension and $\widetilde{\varrho\imath}$ is the corestriction of $\varrho\imath$. Since the downward maps are topological isomorphisms and E' is an extension of topological abelian groups, ϱE is an extension of topological abelian groups, too. $\square$

A special case of Proposition 3.10 for a metrizable group H was proved in [18, Theorem 2].

Proposition 3.11. *Let G be a topological abelian group and H a Čech-complete topological abelian group. If* $\mathrm{Ext}(\varrho G, H)$ *is trivial, then* $\mathrm{Ext}(G, H)$ *is trivial as well.*

Proof. Suppose that $\mathrm{Ext}(\varrho G, H)$ is trivial and consider an extension $E : 1 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$. By Proposition 3.10, $\varrho E : 1 \to H \xrightarrow{\varrho\imath} \varrho X \xrightarrow{\varrho\pi} \varrho G \to 1$ is an extension of topological abelian groups. By hypothesis, ϱE splits, so by Theorem 1.2 there exists a continuous homomorphism $P\colon \varrho X \to H$ such that $P \circ \varrho\imath = \mathrm{Id}_H$. The continuous homomorphism $P|_X\colon X \to H$ satisfies $P|_X \circ \imath = \mathrm{Id}_H$. Hence E splits. $\square$

Proposition 3.12. *Let G be a topological abelian group and $\{H_i : i \in I\}$ a family of topological abelian groups. Then* $\mathrm{Ext}(G, H_i)$ *is trivial for every $i \in I$ if and only if* $\mathrm{Ext}(G, \prod_{i\in I} H_i)$ *is trivial.*

Proof. For every $j \in I$, consider the projections $p_j\colon \prod_{i\in I} H_i \to H_j$ and inclusions $q_j\colon H_j \to \prod_{i\in I} H_i$. Suppose that $\mathrm{Ext}(G, H_i)$ is trivial for every $i \in I$. Let $1 \to \prod_{i\in I} H_i \xrightarrow{\imath} X \xrightarrow{\pi} G \to 1$ be an extension of topological abelian groups. For every $j \in I$, using Lemma 1.1, we can construct a diagram in the following way

$$\begin{tikzcd}
1 \arrow[r] & \prod_{i\in I} H_i \arrow[r, "\imath"] \arrow[d, "p_j"] & X \arrow[r, "\pi"] \arrow[d, "s_j"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H_j \arrow[r, "r_j"] & PO_j \arrow[r, "p"] & G \arrow[r] & 1
\end{tikzcd}$$

where PO_j, r_j and s_j form the push-out of $\imath$ and p_j, and the bottom sequence is an extension of topological abelian groups. By hypothesis, there exists a continuous homomorphism $t_j\colon PO_j \to H_j$ satisfying $t_j \circ r_j = \mathrm{Id}_{H_j}$. It is clear that the map $x \in X \mapsto ((t_i \circ s_i)(x))_{i\in I} \in \prod_{i\in I} H_i$ is a left inverse for $\imath$.

Suppose now that $\mathrm{Ext}(G, \prod_{i\in I} H_i)$ is trivial. Fix $j \in I$ and let $1 \to H_j \xrightarrow{\imath'} X \xrightarrow{\pi'} G \to 1$ be an extension of topological abelian groups. Again by Lemma 1.1, we can consider a diagram

$$\begin{tikzcd}
1 \arrow[r] & H_j \arrow[r, "\imath'"] \arrow[d, "q_j"] & X \arrow[r, "\pi'"] \arrow[d, "s"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \prod_{i\in I} H_i \arrow[r, "r"] & PO'_j \arrow[r, "p'"] & G \arrow[r] & 1
\end{tikzcd}$$

where PO'_j, r and s form the push-out of $\imath'$ and q_j and the bottom sequence is an extension of topological abelian groups. By hypothesis, there exists a continuous homomorphism $t\colon PO'_j \to \prod_{i\in I} H_i$ with $t \circ r = \mathrm{Id}_{\prod_{i\in I} H_i}$. It is clear that $p_j \circ t \circ s$ is a left inverse for $\imath'$. $\square$

Now we are ready to present the main result of this section.

Theorem 3.13. *Let $G = \prod_{i\in I} G_i$ be the product of a family of topological abelian groups such that each factor G_i is a dense subgroup of a MAP and Čech-complete group. Assume that both $\mathrm{Ext}(G_i, \mathbb{R})$ and $\mathrm{Ext}(G_i, \mathbb{T})$ are trivial for each $i \in I$. If H is an arbitrary product of copies of $\mathbb{R}$ and $\mathbb{T}$, then $\mathrm{Ext}(G, H)$ is trivial.*

Proof. In view of Proposition 3.12, it suffices to show that $\mathrm{Ext}(G, M)$ is trivial when M is either $\mathbb{R}$ or $\mathbb{T}$. Since G_i is a dense subgroup of a MAP and Čech-complete group L_i, the group $\varrho G \cong \prod_{i\in I} \varrho G_i \cong \prod_{i\in I} L_i$ is a product of Čech-complete groups. By Theorem 3.8, we have $\mathrm{Ext}(\varrho G, M)$ is trivial. It now follows from Proposition 3.11 that $\mathrm{Ext}(G, M)$ is trivial. $\square$

It is clear that locally precompact groups are dense subgroups of locally compact groups which are obviously MAP and Čech-complete. By Proposition 1.6, both $\mathrm{Ext}(L, \mathbb{T})$ and $\mathrm{Ext}(L, \mathbb{R})$ are trivial for each locally compact abelian group L. Hence the next corollary follows from Theorem 3.13.

Corollary 3.14. *Let $G = \prod_{i\in I} G_i$ be the product of a family of locally precompact abelian groups. If H is an arbitrary product of copies of $\mathbb{R}$ and $\mathbb{T}$, then $\mathrm{Ext}(G, H)$ is trivial.*

Our last result is a special case of Corollary 3.14 which generalizes Theorem 1(a) in [7].

Corollary 3.15. *Let $G = \prod_{i\in I} G_i$ be the product of a family of locally precompact abelian groups. If H is either $\mathbb{R}$ or $\mathbb{T}$, then $\mathrm{Ext}(G, H)$ is trivial.*

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