Facultad de Ciencias Económicas y Empresariales
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Luis Alberiko Gil-Alana

Facultad de Ciencias Económicas y Empresariales
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ABSTRACT

Multivariate tests of fractionally integrated hypotheses are proposed in this article. They are a natural generalization of the univariate tests of Robinson (1994) for testing unit roots and other nonstationary hypotheses. The functional forms of the tests, based on the score principle are calculated in both, the time and the frequency domain. Some simulations based on Monte Carlo experiments and a small empirical application are also carried out at the end of the article.

Luis A. Gil-Alana
Universidad de Navarra
Departamento de Economía
31080 Pamplona
SPAIN
alana@unav.es

1. Introduction

Time series data occur commonly in the natural and engineering sciences, economics and many other fields of enquiry. A typical feature of such data is their apparent dependence across time, for example sometimes records close together in time are strongly correlated. A general practice in economics is to model the nonstationary character of the series in terms of unit root models, which typically assume that the first differences are stationary. Many test statistics have been developed for this purpose (see, e.g., Fuller, 1976; Dickey and Fuller, 1979; Schmidt and Phillips, 1992; etc.). However, most of these tests are embedded in autoregressive (AR) alternatives that are stationary or explosive. The test statistics often have nonstandard null and local asymptotic distributions. The AR model, however, is merely one of the many models that nest a unit root. Robinson (1994) proposed a very general testing procedure which includes, as a particular case, the testing of a unit root embedded in fractional alternatives of form

$$(1 - L)^d \, x_t \; = \; u_t, \qquad t = 1, 2, ... \tag{1}$$

where L is the lag operator ($Lx_t = x_{t-1}$); $\{u_t\}$ is a covariance stationary sequence with zero mean and weak parametric autocorrelation; d can be any real number, and the unit root case corresponds to the null d = 1. Processes like (1) with positive non-integer d are called fractionally integrated and when u_t is ARMA(p, q),, x_t has been called a fractionally ARIMA (ARFIMA) (p, d, q) process. These models were introduced by Granger and Joyeux (1980), Granger (1980, 1981) and Hosking (1981), (although earlier work by Adenstedt, 1974 and Taqqu, 1975, shows an awareness of the representation), and were theoretically justified, in terms of aggregation by Robinson (1978) and Granger (1980).

We propose, in this article, multivariate tests for unit roots and other fractionally integrated hypotheses, which are a generalization of the univariate tests of Robinson (1994). They are relevant if we want to analyse the interrelationships between different variables, also providing a more detailed insight into properties and stochastic behaviour than the univariate work. For example, we might want to investigate the order of integration of a given variable across countries, allowing for weak dependence across the residuals of the differenced series. In doing so, we can determine the degree of persistence of the variable across countries. Also, it might be of interest to examine the degrees of integration of several variables simultaneously when they are specified in a multivariate system and this will be illustrated with an example in Section 6. We can consider a regression model of form

$$Y_t \;=\; Z_t(\delta) \;+\; X_t, \qquad t \;=\; 1, 2, ... \tag{2}$$

$$X_t \;=\; 0, \qquad t \;\leq\; 0, \tag{3}$$

where the column vectors Y_t and X_t each has n components, and by δ we mean a (k x 1) vector of real parameters. $Z_t(\delta)$ is a (n x 1) vector of (possible) non-linear functions of δ and, in general a number of predetermined variables. We assume that under the null hypothesis to be tested and described below, X_t in (2) and (3) satisfies

$$\Phi(L)\, X_t \;=\; U_t, \qquad t \;=\; 1, 2, ..., \tag{4}$$

where Φ(L) is a (n x n) diagonal matrix function of L,[1] and U_t is a (n x 1) vector process (defined as in Robinson, 1995, for example, as a covariance stationary process with spectral density matrix f(λ) that is finite and positive definite), with mean zero and weak parametric autocorrelation. We consider a diagonal matrix function Φ(z; θ) of the complex variate z and the p-dimensional vector θ of real-valued parameters, where Φ(z; θ) = Φ(z) for all z such that |z| = 1 if and only if the null hypothesis defined by

$$H_o: \;\; \theta \;=\; 0 \tag{5}$$

[1] Non-diagonal matrices Φ may also be considered and though this would lead to the possibility of a fractionally cointegrated system (see, e.g., Robinson and Yajima, 2002), its study is out of the scope of this paper.

holds, where there is no loss of generality in using the vector of zeros instead of an arbitrary given vector. Thus, we can cast (4) in terms of a nested composite parametric null hypothesis, within the class of alternatives

$$\Phi(L;\theta)\, X_t \;=\; U_t, \qquad t \;=\; 1, 2, ... \tag{6}$$

We take Φ(z) to have u[th] element: $\rho_u(z) \;=\; (1-z)^{\gamma_1^u}\;(1+z)^{\gamma_2^u}\prod_{j=3}^{h^u}(1-2\cos w_j^u\, z + z^2)^{\gamma_j^u}$

for a given h^u, given distinct real numbers w_j^u, j = 3, 4, …, h^u on the interval (0, π) and given real numbers γ_j^u for j = 1,…,h^u. Thus, (4) permits us to consider a wide range of possibilities. We briefly indicate some special cases of interest:

a): I(1) processes: if $\rho_u(z) = (1-z)$, and in general, I(d) processes, (eg. Gil-Alana and Robinson, 1997; Gil-Alana, 2001a), if $\rho_u(z) = (1-z)^d$.

b): Cyclic I(1) processes: if $\rho_u(z) = (1 - 2\cos wz + z^2)$ for 0 < w < π, and similarly, fractional cyclical models, (Gray et al., 1989, 1994; Gil-Alana, 2001b), if $\rho_u(z) = (1 - 2\cos wz + z^2)^d$.

c): Quarterly I(1) processes: if $\rho_u(z) = (1-z^4)$, and quarterly I(d), (Porter-Hudak, 1990; Gil-Alana and Robinson, 2001), if $\rho_u(z) = (1-z^4)^d$, and so on.

We specify Φ(z; θ) in a way such that we take each element of Φ(z; θ), $\rho_u(z;\theta)$, to depend on θ but not necessarily involving all elements of θ, specifically

$$\rho_u(z;\theta) \;=\; (1-z)^{\gamma_1^u+\theta_{i1}^u}\;(1+z)^{\gamma_2^u+\theta_{i2}^u}\prod_{j=3}^{h^u}(1-2\cos w_j^u\, z + z^2)^{\gamma_j^u+\theta_{ij}^u} \tag{7}$$

where for each combination (u, j), $\theta_{ij}^u = \theta_l$ for some l; and for each l, there is at least one combination (u, j) such that $\theta_{ij}^u = \theta_l$, where θ_l corresponds to the l[th] element of θ. This specification allows us to consider duplications not only within equations but also across equations. Furthermore, this way of specifying Φ(z; θ) permits us to specifically consider situations where θ is the same across all equations, and also the case when θ is taken as strictly different for each equation. This will be illustrated with some examples in Section 4. We adopt the normalization $\rho_u(0; \theta) = 1$ for all θ and u = 1,2,…n, and assume that $\rho_u(z; \theta)$ is differentiable in θ on a neighbourhood of $\theta = 0$ for all $|z| = 1$. Also we suppose that for any u,v = 1, 2, …n

$$\det(E_{uv}) \; < \; \infty, \tag{8}$$

$$E_{uv} \; = \; \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\varepsilon_{(u)}(\lambda)\,\bar{\varepsilon}_{(v)}(\lambda)' + \varepsilon_{(v)}(\lambda)\,\bar{\varepsilon}_{(u)}(\lambda)'\right)d\lambda\,; \qquad \varepsilon_{(u)} \; = \; \frac{\partial \log \rho_u(e^{i\lambda};\theta)}{\partial\theta},$$

for real λ, where $\bar{\varepsilon}_{(u)}(\lambda)$ is the conjugate vector of $\varepsilon_{(u)}(\lambda)$. Note that the (p x 1) vector $\varepsilon_{(u)}(\lambda)$ is independent of θ given the linearity of log $\rho_u(e^{i\lambda}; \theta)$ with respect to θ in (7). In particular, its real part takes the form: $\delta_{1l}^{u}\log\left|2\sin\frac{\lambda}{2}\right| + \delta_{2l}^{u}\log\left(2\cos\frac{\lambda}{2}\right) + \sum_{j=3}^{h^u}\delta_{jl}^{u}\log\left|2(\cos\lambda - \cos w_j^u)\right|$, for $l = 1,\ldots p$ and $|\lambda| < \pi$,

where $\delta_{jl}^{u} = 1$ if $\theta_{ij}^{u} = \theta_l$ and 0 otherwise. (See Robinson, 1994, page 1422). Condition (8) will not be satisfied when testing unit roots embedded in AR alternatives of form: $\rho_u(z; \theta) = (1 - (1+\theta)z)$. However, it will be satisfied if the alternatives are fractional of form: $\rho_u(z; \theta) = (1 - z)^{1+\theta}$.

Under the null hypothesis (5), the model in (2) – (4) can be redefined as

$$\Phi(L)\,Y_t \; = \; W_t(\delta) + U_t, \quad t \; = \; 1, 2, ... \tag{9}$$

where $W_t(\delta) = (W_{1t}(\delta); W_{2t}(\delta); \ldots; W_{nt}(\delta))'$, with $W_{ut}(\delta) = \rho_u(L)\, Z_{ut}(\delta)$. (9) is a very general form of a regression model, which includes multivariate linear and non-linear models and simultaneous equations systems. Sections 2 and 3 present the functional forms of the test statistics for the cases of white noise and weakly autocorrelated U_t. In section 4, the tests are rewritten for two cases of interest: First, we suppose that θ in (6) is the same across all elements in $\Phi(z;\theta)$. Then, we take θ as strictly different for each element in $\Phi(z;\theta)$. Section 5 reports some simulations, studying the finite-sample behaviour of versions of the tests. Section 6 contains a small empirical application while Section 7 concludes. Appendices A and B show the derivations of the test statistics of sections 2 and 3 respectively.

2. Score tests for white noise U_t

We describe a score test for H_o (5) in a model given by (2), (3) and (6), under the presumption that U_t in (6) is a vector sequence of zero mean uncorrelated in time random variables, with unknown variance-covariance matrix K. One definition for the score test is as follows. Let $L(\eta)$ be an objective function and take $\eta = (\theta', \nu')'$, where $\tilde{\eta} = (0', \tilde{\upsilon}')'$ are the values that minimizes $L(\eta)$ under the null. A score test (see Rao,1973, page 418) is then given by

$$\frac{\partial L(\eta)}{\partial \eta'}\left[E_o\left(\frac{\partial L(\eta)}{\partial \eta}\frac{\partial L(\eta)}{\partial \eta'}\right)\right]^{-1}\frac{\partial L(\eta)}{\partial \eta}\bigg|_{\theta = 0;\, \nu = \tilde{\nu}}, \qquad (10)$$

where the expectation is taken under the null prior to substitution of $\tilde{\nu}$. The same asymptotic behaviour will be expected, however, if we replace the inverted matrix in (10) by alternative forms such as the sample average or the Hessian, (see, eg. Godfrey, 1988). For convenience below, we make use of the expected information matrix, so the score test takes the form:

$$\frac{\partial L(\eta)}{\partial \eta'}\left[E_o\left(\frac{\partial^2 L(\eta)}{\partial \eta\, \partial \eta'}\right)\right]^{-1}\frac{\partial L(\eta)}{\partial \eta}\bigg|_{\theta = 0;\, \nu = \tilde{\nu}} \qquad (11)$$

We take L in (11) to be the negative of the log-likelihood based on Gaussian U_t, with $\eta = (\theta', \delta', \alpha')'$; $\alpha = v(K)$.[2] In Appendix A it is shown that (11) takes the form

$$\hat{S}^t = T\, \hat{a}^{t\,\prime}\, (\hat{A}^t)^{-1}\, \hat{a}^t \qquad (12)$$

where $\hat{a}^t$ is a (p x 1) vector of form

$$\hat{a}^t \;=\; -\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\sum_{s=1}^{T-1}\psi_s^{(u)}\, C_{uv}(s;\hat{\delta}), \tag{13}$$

and $\psi_s^{(u)}$ is obtained by expanding $\psi_{(u)}(\lambda) = \mathrm{Re}[\varepsilon_{(u)}(\lambda)]$ *as* $\sum_{s=1}^{\infty}\psi_s^{(u)}\cos\lambda\, s.$

$$\hat{A}^t \;=\; \sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\,\hat{\sigma}_{uv}\sum_{s=1}^{T-1}\left(1-\frac{s}{T}\right)\psi_s^{(u)}\,\psi_s^{(v)}{}', \tag{14}$$

$\hat{\sigma}^{uv}$ is the (u,v)th element of $\hat{K}^{-1}$; $\hat{\sigma}_{uv}$ is the (u,v)th element of $\hat{K}$; and $C_{uv}(s;\hat{\delta})$ is the (u,v)th element of $C_{\hat{U}}(s)$, where $\hat{K} = \frac{1}{T}\sum_{t=1}^{T}\hat{U}_t(\delta)\hat{U}_t(\delta)'$; $C_{\hat{U}}(s) = \frac{1}{T}\sum_{t=1}^{T-s}\hat{U}_t(\delta)\hat{U}_{t+s}(\delta)'$; $\hat{U}_t(\delta)$ $= \Phi(L)Y_t - W_t(\hat{\delta})$ and $\hat{\delta}$ must be at least a $T^{1/2}$-consistent estimate of the true value δ.

As in the univariate tests of Robinson (1994), concise formulas for $\psi_s^{(u)}$ are available in some simple cases; for example, $\psi_s^{(u)} = -s^{-1}$, when $\rho_u(L;\theta) = (1-L)^{d+\theta}$. However, we can also express the test statistic in the frequency domain and, under certain suitable conditions, (basically a generalization of those in Robinson, 1994, requiring technical assumptions on ρ_u, and thus, on $\varepsilon_{(u)}(\lambda)$, to justify approximating integrals by sums), approximate this to obtain an alternative, asymptotically equivalent, form. Thus, (13) can be written as

$$\hat{a}^t \;=\; -\frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\int_{-\pi}^{\pi}\left(\varepsilon_{(u)}(\lambda)+\bar{\varepsilon}_{(v)}(\lambda)\right)I_{uv}(\lambda;\hat{\delta})\,d\lambda$$

where $\varepsilon_{(u)}(\lambda)$ is given below (8); $\bar{\varepsilon}_{(v)}(\lambda)$ is the conjugate vector of $\varepsilon_{(v)}(\lambda)$, and $I_{uv}(\lambda;\hat{\delta})$ is the (u,v)th element in the cross-periodogram of $\hat{U}_t(\delta) = (\hat{U}_{1t}(\delta); \ldots; \hat{U}_{nt}(\delta))'$:

$$I_{uv}(\lambda;\hat{\delta}) \;=\; W_u(\lambda;\hat{\delta})\overline{W}_v(\lambda;\hat{\delta}), \qquad W_u(\lambda;\hat{\delta}) \;=\; \frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T}\hat{U}_{ut}(\delta)e^{i\lambda t},$$

[2] v(K) means a vector containing the columns of K, including only those non-repetitive elements of K.

where the line over W denotes complex conjugate. Also, under suitable conditions, keeping $\hat{\sigma}^{uv}$ and $\hat{\sigma}_{uv}$ fixed, $\hat{A}^t$ in (14) becomes asymptotically

$$\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\,\hat{\sigma}_{uv}\sum_{s=1}^{\infty}\psi_s^{(u)}\,\psi_s^{(v)}{}', \qquad (15)$$

and using the Parseval's relationship (see, Zygmund, 1979, Chapter 4), this quantity can be expressed as

$$\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\hat{\sigma}_{uv}\,\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\varepsilon_{(u)}(\lambda)\,\bar{\varepsilon}_{(v)}(\lambda)' + \varepsilon_{(v)}(\lambda)\,\bar{\varepsilon}_{(u)}(\lambda)'\right)d\lambda \;=\; \sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\hat{\sigma}_{uv}\,E_{uv}\,.$$

Therefore, the score statistic in (12) can be approximated in the frequency domain by the expression

$$\hat{S}^f \;=\; T\;\hat{a}^f{}'(\hat{A}^f)^{-1}\,\hat{a}^f \qquad (16)$$

where
$$\hat{a}^f \;=\; \frac{-\pi}{T}\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\sum_{r}^{*}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)I_{uv}(\lambda_r;\hat{\delta}), \qquad (17)$$

$$\hat{A}^f \;=\; \frac{1}{2T}\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\hat{\sigma}_{uv}\sum_{r}^{*}\left(\varepsilon_{(u)}(\lambda_r)\,\bar{\varepsilon}_{(v)}(\lambda_r)' + \varepsilon_{(v)}(\lambda_r)\,\bar{\varepsilon}_{(u)}(\lambda_r)'\right), \qquad (18)$$

$\lambda_r = 2\pi r/T$, the sums on the asterisk being over λ_r in M where $M = \{\lambda;\ -\pi < \lambda < \pi;\ \lambda \notin (\rho_l - \lambda;\ \rho_l + \lambda),\ l = 1,2,\ldots,s\}$, such that ρ_l, $l = 1,2,\ldots s$ are the distinct poles on $\varepsilon_{(u)}(\lambda)$ on $(-\pi, \pi]$ for $u = 1,2,\ldots n$. Note that if $\rho_u(L;\theta)$ is given by $(1 - L)^{d+\theta}$, we calculate $\varepsilon_{(u)}(\lambda)$ as:

$$\mathrm{Re}[\varepsilon_{(u)}(\lambda_r)] \;=\; \psi_{(u)}(\lambda_r) \;=\; \log\left|2\sin\frac{\lambda_r}{2}\right|, \quad \textit{and} \quad \mathrm{Im}[\varepsilon_{(u)}(\lambda_r)] \;=\; \frac{\lambda_r - \pi}{2},$$

with $r = 1,2,\ldots,T-1$, (Zygmund, 1979, page 5).

Under some regularity conditions, the test described below will have the same optimal asymptotic properties as Robinson's (1994) univariate tests. These conditions impose a martingale difference assumption on the white noise vector U_t, (that is, $E(U_t \mid B_{t-1}) = 0$ and $E(U_tU_t' \mid B_{t-1}) = K$, where B_{t-1} is the σ-field of events generated by U_s, $s \le t$); also W as defined in Appendix A must be a positive definite matrix; and $\rho_u(z;\theta)$, $u = 1,2,\ldots n$ must belong to the class H as defined in Robinson (1994), with $\varepsilon_{(u)}(\lambda)$ satisfying the same technical conditions as $\psi(\lambda)$ in that paper. Then, (12) and (16) will have a null limit χ^2_p distribution, and under local alternatives of form H_a: $\theta = \theta_T = \delta T^{-1/2}$, a $\chi^2_p(\nu)$ distribution with a non-centrality parameter υ, which is optimal under Gaussianity of U_t. Thus, a large sample 100α%-level test for rejecting H_o (5) against the alternative: H_1: $\theta \neq 0$, will be given by the rule: "Reject H_o if $\hat{S}^t$ $(or\ \hat{S}^f) > \chi^2_{p,\alpha}$", where Prob($\chi^2_p > \chi^2_{p,\alpha}$) = α.

3. Score test for weakly parametrically correlated U_t

We consider the model in (2), (3) and (6), with U_t in (6) as a vector process with n components generated by a parametric model of form

$$U_t = \sum_{j=0}^{\infty} A(j;\tau)\,\varepsilon_{t-j}\,, \qquad t = 1,2,..., \tag{19}$$

where ε_t is a white noise vector process, and K is now the unknown variance-covariance matrix of ε_t. The spectral density matrix of U_t in (19) is

$$f(\lambda;\tau) = \frac{1}{2\pi} k(\lambda;\tau)\, K\, k(\lambda;\tau)^*, \tag{20}$$

where $k(\lambda;\tau) = \sum_{j=0}^{\infty} A(j;\tau) e^{i\lambda j}$ and k^* means the complex conjugate transpose of k. A number of conditions are required on A and f in Appendix B when deriving the test statistic; their practical implications being that though U_t is capable of exhibiting a much stronger degree of autocorrelation than multiple ARMA processes, its spectral density matrix must be finite, with eigenvalues bounded and bounded away from zero. By extending the arguments in Section 2 and Appendix A, we show in Appendix B that, under Gaussianity of U_t, an approximate score statistic for testing (5) in (2), (3), (6) and (19) is

$$\tilde{S} = T\, \tilde{b}'\, \tilde{B}^{-1}\, \tilde{b}, \tag{21}$$

$$\tilde{b} \;=\; \frac{-1}{2\pi}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right) I_{uv}(\lambda_r;\hat{\delta})\,\hat{f}^{vu}(\lambda_r;\tilde{\tau}), \qquad (22)$$

and $\tilde{B} = \tilde{C} - \tilde{D}'\tilde{E}^{-1}\tilde{D}$, where

$$\tilde{C} \;=\; \frac{1}{2T}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r)\bar{\varepsilon}_{(v)}(\lambda_r)' + \bar{\varepsilon}_{(v)}(\lambda_r)\varepsilon_{u)}(\lambda_r)'\right)\hat{f}_{uv}(\lambda_r;\tilde{\tau})\,\hat{f}^{vu}(\lambda_r;\tilde{\tau}), \qquad (23)$$

$$\tilde{D} = \frac{-1}{2T}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)'\right)\hat{f}^{vu}(\lambda_r;\tilde{\tau})\frac{\partial \hat{f}_{uv}(\lambda_r;\tilde{\tau})}{\partial \tau'}, \qquad (24)$$

$$\left(\tilde{E}\right)_{uv} \;=\; \frac{1}{2T}\sum_{r}^{*} tr\left(\hat{f}^{-1}(\lambda_r;\tilde{\tau})\frac{\partial \hat{f}(\lambda_r;\tilde{\tau})}{\partial \tau_u}\hat{f}^{-1}(\lambda_r;\tilde{\tau})\frac{\partial \hat{f}(\lambda_r;\tilde{\tau})}{\partial \tau_v}\right); \qquad (25)$$

$I_{uv}(\lambda;\hat{\delta})$ is the (u,v)th element of the periodogram of $\hat{U}_t$, $I_U(\lambda;\hat{\delta})$, as it was given in Section 2; $\hat{f}_{uv}(\lambda_r;\tilde{\tau})$ and $\hat{f}^{uv}(\lambda_r;\tilde{\tau})$ correspond respectively to the (u,v)th elements of $\hat{f}(\lambda_r;\tilde{\tau})$ and $\hat{f}^{-1}(\lambda_r;\tilde{\tau})$, with $\hat{f}(\lambda;\tilde{\tau}) = (1/2\pi)k(\lambda;\tilde{\tau})\hat{K}\,k(\lambda;\tilde{\tau})^*$ and

$$\tilde{\tau} \;=\; \arg\min_{\tau\in T^*}\left(\frac{T}{2}\log\det \hat{f}(\lambda_r;\tau) + \frac{1}{2}\sum_{r}^{*} tr\left[\hat{f}^{-1}(\lambda_r;\tau)I_U(\lambda_r;\hat{\delta})\right]\right),$$

where T^* is a compact subset of q-dimensional Euclidean space.

Extending the conditions in Robinson (1994) and thus, allowing for a martingale difference assumption on ε_t in (19), with $\sum_{j=1}^{\infty} j^{1/2}\| A(j;\tau) \| < \infty$, where $\| A \|$ means any norm for the matrix A, for example, the square root of the maximum eigenvalue of A^*A; with W as a positive definite matrix; ρ_u, u = 1,…,n, satisfying the same conditions as in Section 2; and $f_{uv}(\lambda;\tau)$ and $\partial f_{uv}(\lambda;\tau)/\partial\tau$ satisfying a Lipschitz condition in λ of order η > ½, for all u,v = 1,2,..n, (Hannan, 1970, page 513), then, under H_o (5): $\tilde{S} \rightarrow_d \chi^2_p$ as T → ∞, and $\tilde{S}$ should also satisfy the same asymptotic efficiency properties as $\hat{S}^t$ and $\hat{S}^f$ in Section 2.

4. Particular cases of the score tests

In this section we consider two special cases of interest of each of the previous versions of the tests. The first corresponds to (6) with θ containing the same (p x 1) vector across all elements in Φ(z;θ), whilst the second case takes this vector as strictly different for each equation.

We illustrate both cases with two simple examples in a bivariate model. First we test if one of the series is $I(d_1)$ and the other is $I(d_2)$. Thus, we consider that both series have a root at the same zero frequency, though with different integration orders. In the second example, the series might differ in the number of roots in its bivariate representation, and we test for an $I(d_1)$ process in the first series and a quarterly $I(d_2)$ in the second one. Therefore, the models will be specified under the null, in the first example as

$$\begin{pmatrix} (1-L)^{d_1} & 0 \\ 0 & (1-L)^{d_2} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} \qquad t = 1,2,... \qquad \text{(E1)}$$

and in the second as

$$\begin{pmatrix} (1-L)^{d_1} & 0 \\ 0 & (1-L^4)^{d_2} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} \qquad t = 1,2,... \qquad \text{(E2)}$$

where d_1 and d_2 are given real numbers, (non-necessarily constrained to be in the stationary region); $X_t = (X_{1t}, X_{2t})' = 0$ for $t \leq 0$; and $U_t = (U_{1t}, U_{2t})'$ is an I(0) process.

4.a Same θ across the equations

We consider (2), (3) and (6), and Φ(z; θ) with u^{th} diagonal element:

$$\rho_u(z;\theta) = (1-z)^{\gamma_1^u + \theta_{i_1}} (1+z)^{\gamma_2^u + \theta_{i_2}} \prod_{j=3}^{h^u} (1 - 2\cos w_j^u \, z + z^2)^{\gamma_j^u + \theta_{i_j}},$$

and for each j, $\theta_{i_j} = \theta_l$ for some l, and for each l, there is at least one j such that $\theta_{i_j} = \theta_l$. Therefore, the parameter vector θ is exactly the same across all equations in (6), and the difference between one equation and another comes now through the coefficients γ_j^u for j = 1, 2, ….h^u and u = 1,2,…n. Thus, in the first example, the model will be specified as

$$\begin{pmatrix} (1-L)^{d_1+\theta} & 0 \\ 0 & (1-L)^{d_2+\theta} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} \qquad t = 1,2,...$$

testing the null H_o: θ = 0 against the alternative H_a: θ ≠ 0. In the second example, the model will take the form

$$\begin{pmatrix} (1-L)^{d_1+\theta_1}(1+L)^{\theta_2}(1+L^2)^{\theta_3} & 0 \\ 0 & (1-L)^{d_2+\theta_1}(1+L)^{d_2+\theta_2}(1+L^2)^{d_2+\theta_3} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} \qquad t = 1,2,...$$

which, under the null hypothesis, H_o: θ = (θ_1, θ_2, θ_3)' = 0, becomes (E2), implying that X_{2t} follows a quarterly I(d_2) process.

This specification is a particular case of the general model presented in Section 1 with

$$\varepsilon_{(u)}(\lambda) \;=\; \frac{\partial \log \rho_u(e^{i\lambda};\theta)}{\partial\theta} \;=\; \varepsilon(\lambda) \quad \textit{for all} \;\; u = 1,2,...,n. \qquad (26)$$

(26) implies that $\psi_s^{(u)} = \psi_s$ for all u = 1, 2, …, n. Thus, we can easily describe the functional forms of the three test statistics. Starting with white noise U_t, substituting (26) in (12) – (14), the time domain version of the test statistic is

$$\hat{S}^t \;=\; T\;\hat{a}^{t_1}{}'\;(\hat{A}^{t_1})^{-1}\;\hat{a}^{t_1} \qquad (27)$$

$$\hat{a}^{t_1} \;=\; -\sum_{s=1}^{T-1}\psi_s\, tr[\hat{K}^{-1}C_{\hat{U}}(s)]; \quad \hat{A}^{t_1} \;=\; n\sum_{s=1}^{T-1}\left(1-\frac{s}{T}\right)\psi_s\,\psi_s{}'.$$

Expressing the test in the frequency domain,

$$\hat{S}^{f_1} \;=\; T\;\hat{a}^{f_1}{}'\;(\hat{A}^{f_1})^{-1}\;\hat{a}^{f_1} \qquad (28)$$

$$\hat{a}^{f_1} \;=\; \frac{-2\pi}{T}\sum_r^{*}\psi(\lambda_r)\, tr\left[\hat{K}^{-1}I_U(\lambda_r;\hat{\delta})\right]; \quad \hat{A}^{f_1} \;=\frac{n}{T},$$

and finally, substituting (26) in (21), the test statistic with autocorrelated U_t becomes:

$$\tilde{S}^1 = T\tilde{b}^{1\prime}\left(\tilde{C}^1 - \tilde{D}^{1\prime}\tilde{E}^{-1}\tilde{D}^1\right)^{-1}\tilde{b}^1 \tag{29}$$

$$\tilde{b}^1 = \frac{-1}{T}\sum_r^* \psi(\lambda_r)\, tr\left[I_U(\lambda_r;\hat{\delta})\,\hat{f}(\lambda_r;\tilde{\tau})^{-1}\right]; \qquad \tilde{C}^1 = \frac{2n}{T}\sum_r^* \psi(\lambda_r)\psi(\lambda_r)',$$

$$\tilde{D}^{1\prime} = \frac{-1}{T}\sum_r^* \psi(\lambda_r)\left[tr\left(\hat{f}^{-1}(\lambda_r;\tilde{\tau})\frac{\partial \hat{f}(\lambda_r;\tilde{\tau})}{\partial \tau_1}\right);\ldots;tr\left(\hat{f}^{-1}(\lambda_r;\tilde{\tau})\frac{\partial \hat{f}(\lambda_r;\tilde{\tau})}{\partial \tau_1}\right)\right]$$

$$\tilde{E}_{uv} = \frac{1}{2T}\sum_r^* tr\left[\hat{f}(\lambda_r;\tilde{\tau})^{-1}\frac{\partial \hat{f}(\lambda_r;\tilde{\tau})}{\partial \tau_u}\hat{f}(\lambda_r;\tilde{\tau})^{-1}\frac{\partial \hat{f}(\lambda_r;\tilde{\tau})}{\partial \tau_v}\right].$$

4b. Different θ's across equations

We take θ in (6) to be equal to $(\theta^{1\prime}, \theta^{2\prime}, \ldots, \theta^{n\prime})'$, where θ^u is a (p_u x 1) vector affecting only to the u^{th} equation. Thus, the parameter vector involving θ will be strictly different for each equation. We take the u^{th} element of $\Phi(z;\theta)$, $\rho_u(z;\theta^u)$, adopting the same functional form as in (7), where now for each j, $\theta^u_{ij} = \theta^u_l$ for some l, and for each l, there is at least one j such that $\theta^u_{ij} = \theta^u_l$. Thus, in the first example, the model is

$$\begin{pmatrix}(1-L)^{d_1+\theta^1_1} & 0 \\ 0 & (1-L)^{d_2+\theta^2_1}\end{pmatrix}\begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix} = \begin{pmatrix}U_{1t}\\ U_{2t}\end{pmatrix} \qquad t = 1,2,\ldots$$

with $\theta = (\theta^{1\prime}, \theta^{2\prime})' = \left(\theta^1_1;\, \theta^2_1\right)'$, and in the second example,

$$\begin{pmatrix}(1-L)^{d_1+\theta^1_1}(1+L)^{\theta^1_2}(1+L^2)^{\theta^1_3} & 0 \\ 0 & (1-L\,)^{d_2+\theta^2_1}(1+L)^{d_2+\theta^2_2}(1+L^2)^{d_3+\theta^2_3}\end{pmatrix}\begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix} = \begin{pmatrix}U_{1t}\\ U_{2t}\end{pmatrix} \qquad t = 1,2,\ldots$$

with $\theta = (\theta^{1\prime}, \theta^{2\prime})' = \left(\theta^1_1;\, \theta^1_2;\, \theta^1_3;\, \theta^2_1;\theta^2_2;\, \theta^2_3\,\right)'$.

Again this specification is a particular case of the general model presented in Section 1. We need to define the (p_u x 1) vectors $e_{(u)}(\lambda) = \dfrac{\partial \log \rho_u(e^{i\lambda};\theta^u)}{\partial \theta^u}$ and $f_{(u)}(\lambda) = \mathrm{Re}[e_{(u)}(\lambda)]$, for all u = 1, 2, …n, sharing the same properties as $\varepsilon_{(u)}(\lambda)$ and $\psi_{(u)}(\lambda)$ in sections 1 – 3. To show this, we just need to note that $\varepsilon_{(u)}(\lambda) = P_u\, e_{(u)}(\lambda)$, where P_u is a (p x p_u) matrix of 1's and 0's of form $P_u = (0; I_{p_u};\ 0)'$. Substituting $\varepsilon_{(u)}(\lambda)$ in (12), (16) and (21), we can easily obtain the functional forms of the three statistics. Starting with the time domain and white noise U_t, noting that $\psi_s^{(u)} = P_u\, f_s^{(u)}$ where $f_s^{(u)}$ comes from expanding $f_{(u)}(\lambda)$ in terms of its infinite representation, the test statistic takes the form

$$\hat{S}^{t_2} \ = \ T\ \hat{a}^{t_2}{}'(\hat{A}^{t_2})^{-1}\, \hat{a}^{t_2} \tag{30}$$

$$\hat{a}^{t_2} \ = \ (\hat{a}_1^{t_2}{}';\hat{a}_2^{t_2}{}';...;\hat{a}_n^{t_2}{}')'; \quad \hat{a}_u^{t_2} \ = \ -\sum_{v=1}^{n}\hat{\sigma}^{uv}\sum_{s=1}^{T-1}C_{uv}(s;\hat{\delta})\, f_s^{(u)},$$

$$\hat{A}^{t_2} \ = \ \left(\hat{a}_{uv}^{t}\right)_{uv}; \quad \hat{a}_{uv}^{t} \ = \ \hat{\sigma}^{uv}\,\hat{\sigma}_{uv}\sum_{s=1}^{T-1}\left(1-\frac{s}{T}\right) f_s^{(u)}\, f_s^{(v)}{}'.$$

The test statistic in the frequency domain becomes

$$\hat{S}^{f_2} \ = \ T\ \hat{a}^{f_2}{}'(\hat{A}^{f_2})^{-1}\, \hat{a}^{f_2} \tag{31}$$

$$\hat{a}^{f_2} \ = \ (\hat{a}_1^{f_2}{}';\hat{a}_2^{f_2}{}';...;\hat{a}_n^{f_2}{}')'; \quad \hat{a}_u^{f_2} \ = \ -\frac{2\pi}{T}\sum_{v=1}^{n}\hat{\sigma}^{uv}\sum_{r}^{*} e_{(u)}(\lambda_r)\, I_{uv}(\lambda_r;\hat{\delta}),$$

$$\hat{A}^{f_2} \ = \ \left(\hat{a}_{uv}^{f}\right)_{uv}; \quad \hat{a}_{uv}^{f} \ = \frac{2}{T}\hat{\sigma}^{uv}\,\hat{\sigma}_{uv}\sum_{r}^{*} f_{(u)}(\lambda_r)\, f_{(v)}(\lambda_r)';$$

and allowing autocorrelated U_t,

$$\widetilde{S}^{2} \ = \ T\ \widetilde{b}^{2}{}'\left(\widetilde{C}^{2} - \widetilde{D}^{2}{}'(\widetilde{E})^{-1}\,\widetilde{D}^{2}\right)^{-1}\widetilde{b}^{2} \tag{32}$$

$$\widetilde{b}^{2} \ = \ \left(\widetilde{b}_1^{2}{}';\widetilde{b}_2^{2}{}';...;\widetilde{b}_n^{2}{}'\right)'$$

$$\text{with} \quad \widetilde{b}_u^{2} \ = \ -\frac{1}{T}\mathrm{Re}\left(\sum_{r}^{*} e_{(u)}(\lambda)\sum_{v=1}^{n} I_{uv}(\lambda_r;\hat{\delta})\,\hat{f}^{vu}(\lambda_r;\widetilde{\tau})\right),$$

$$\widetilde{C}^{2} \ = \ \left(\widetilde{c}_{uv}\right)_{uv} \quad \text{with} \quad \widetilde{c}_{uv} \ = \ \frac{1}{T}\mathrm{Re}\left(\sum_{r}^{*} e_{(u)}(\lambda_r)\,\overline{e}_{(v)}(\lambda_r)'\,\hat{f}_{uv}(\lambda_r;\widetilde{\tau})\,\hat{f}^{vu}(\lambda_r;\widetilde{\tau})\right),$$

$$\widetilde{D}^{2}{}' \ = \ \left(\widetilde{D}_1;\,\widetilde{D}_2;...;\widetilde{D}_n\right)' \quad \text{with} \quad \widetilde{D}_u{}' \ = \ -\frac{1}{T}\mathrm{Re}\left(\sum_{r}^{*} e_{(u)}(\lambda_r)\sum_{v=1}^{n}\hat{f}^{uv}(\lambda_r;\widetilde{\tau})\frac{\partial \hat{f}_{vu}(\lambda_r;\widetilde{\tau})}{\partial\,\tau'}\right)$$

and $\widetilde{E}$ remains unchanged, i.e., as in (29) and below.

5. Finite sample performance

This section examines the finite-sample behaviour of versions of the above statistics by means of Monte Carlo simulations. All calculations were carried out using Fortran and the NAG's library random number generator on LSE's VAX computer. Given the variety of tests and the number of possibilities covered by them, we concentrate on a bivariate system where the null hypothesis consists of two series following a random walk. We take the model

$$\begin{pmatrix} (1-L)^{1+\theta_1} & 0 \\ 0 & (1-L)^{1+\theta_2} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} \qquad t \;=\; 1, 2, .., T \qquad (33)$$

and $X_t = (X_{1t}, X_{2t})' = 0$ for all $t \leq 0$. Under the null hypothesis:

$$H_o : \theta \;=\; (\theta_1, \theta_2)' \;=\; 0, \qquad (34)$$

$U_t \;=\; (U_{1t}, U_{2t})'$ will be initially, a white noise vector process with mean zero and variance-covariance matrix Σ. First, and without loss of generality, we assume that $\Sigma = I_2$, though we also present results, for a given positive definite matrix Σ. We look at the rejection frequencies of the score statistic given by (30), for fractional alternatives with $(\theta_i)_{i\,=\,1,2}$ in (33) equal to: -0.8, (0.2), 0.8. Then, we generate Gaussian series for different sample sizes, (T = 50, 100 and 200) taking 5000 replications of each case, and present results for a nominal size of 5%.

Table 1 reports rejection frequencies of $\hat{S}^{t_2}$ in (30) with $\Sigma = I_2$. We see that the size of the test is too small in all cases, though it improves as we increase the number of observations. Thus, for example, we see that the size is 1.2% with T = 50; it increases to 2.0% with T = 100, and becomes 3.2% with T = 200. If we concentrate on small departures from (34), we observe that these rejection frequencies increase strongly, especially when the sample size is large. This increase is more important when θ_1 and θ_2 take the same value, though also is noticeable when $\theta_1 \neq \theta_2$. If T = 200, we see that the lowest rejection probability, apart from that of the true model ($\theta_1 = \theta_2 = 0$) is 0.671, which is obtained when $\theta_1 = 0$ and $\theta_2 = -0.2$. However, if $\theta_1 = \theta_2 = -0.2$, it becomes 0.997. Another remarkable feature observed in this table is the fact that when the sample size is small, there is a bias toward positive values of θ_1 and θ_2, though increasing the sample size, the bias tends to disappear.

(Tables 1 and 2 about here)

Table 2 reports rejection frequencies of the same statistic as in Table 1 but taking Σ as a positive definite matrix of form: [(1, 1)'; (1, 2)']. Thus, we can see if the test statistic is robust to a different specification of the variance-covariance matrix of the differenced series. We observe that the size is slightly greater than before, but again too small with respect to the nominal one, though increasing with T. A bias for positive values of θ_1 and θ_2 is again observed when the sample size is small. Comparing the results here with those in Table 1, we see that in most of the cases, the rejection frequencies are now slightly greater, but in general, the results are similar across both tables, suggesting that the test statistic is not much affected by different specifications of the variance-covariance matrix Σ.

Tables 3 and 4 present the empirical sizes of the test based on the frequency domain. Table 3 reports sizes of $\hat{S}^{f_2}$ in (31) assuming first, in Table 3a, that $\Sigma = I_2$ and then, Σ = [(1, 1)'; (1, 2)'] in Table 3b. As in the previous tables, we see that the sizes are very small when T = 50, however, they considerably improve when we increase the number of observations. When $\Sigma \neq I_2$ the same conclusion holds, with empirical sizes smaller than nominal ones but increasing with T. Comparing the empirical sizes in this table with those in Tables 1 and 2, we see that they are very similar. If T = 50, the sizes are now slightly smaller than in the time domain version, but when T = 100 or 200, they are slightly greater.

(Tables 3 and 4 about here)

Finally, Table 4 reports sizes of the test statistic $\tilde{S}^2$ in (32), i.e., the frequency domain version with weakly autocorrelated U_t. In Table 4a we assume that U_t follows a VAR(1) representation, and we choose the parameterization

$$\begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} U_{1t-1} \\ U_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \qquad (35)$$

where $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ is normally distributed with mean zero and variance-covariance matrix I_2. In Table 4b we consider a VMA(1) structure on U_t, using the same parameters as in the VAR(1) case. That is,

$$\begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} + \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.5 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \end{pmatrix}, \qquad (36)$$

and again ε_t being normally distributed with mean 0 and variance I_2.

In both tables we see that the sizes are too large for all nominal sizes, especially when T = 50, however, increasing T, these empirical sizes reduce and they tend to approximate to the nominal values. Thus, for the VAR(1) case, we see that if the number of observations is 200, the sizes are 10.4% for α = 10%; 6.0% for α = 5%; 3.1% for α = 2.5%; and 1.2% for α = 1%. When the VMA(1) structure is considered, the empirical sizes are now slightly greater than in the VAR case, but again we observe a considerable improvement with T. Similar results were obtained when we used different parameters in (35) and (36) and a different variance-covariance matrix for the residuals ε_t.

As a conclusion, we can summarise the results obtained across these tables by saying that the score test statistics obtained in Sections 2 – 4 seem to be adequate to test the null hypothesis of a random walk in this bivariate context. Though sizes are, in most of the cases, smaller than nominal ones, the performance of the tests seems quite good even for small departures from the null, especially if the number of observations is large. The FORTRAN codes used to obtain the score statistics can be found in Gil-Alana (1997) and they are available from the author upon request.

6. An empirical application

The data used in this section are US quarterly real per capita consumption on non-durables and real per capita disposable income from 1947.1 to 1981.2. We use these data because they have been widely employed in the literature to examine the cointegrating relationship between consumption and income (see, e.g., Davidson et al., 1978; Hall, 1978; Engle and Granger, 1987; etc.).

We start by specifying the model in a general form, which, in this bivariate set-up may adopt the form:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} 1 \\ t \end{pmatrix} + \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}; \qquad \begin{pmatrix} (1-L)^{d_1+\theta_1} & 0 \\ 0 & (1-L)^{d_2+\theta_1} \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} U_{1t} \\ U_{2t} \end{pmatrix} \qquad t = 1,2,...$$

where $Y_t = (Y_{1t}, Y_{2t})'$ correspond to the original time series (consumption and income) and the null hypothesis is given by (34).

We performed the score statistics in (30) – (32), depending basically on the choice for the disturbance vector U_t and the inclusion or not of restrictions in the elements of the matrix B above, and take values of d_1 and d_2 ranging from 0.6 to 1.4 with 0.1 increments. The test statistic reported in Table 5 is the one corresponding to $\hat{S}^{f_2}$ in (31), i.e., the frequency domain version of the test for white noise U_t. We also performed the test in the time domain and the results were very similar to those given in this table. We initially impose that B = 0 a priori and the results are given in the upper part of Table 5. We see that the only non-rejection value takes place when $d_1 = d_2 = 0.9$, and any departure from this case strongly increases the value of the test statistic. Next, we consider the cases of $B_{12} = B_{22} = 0$ a priori, (with $Z_t \equiv 1$), i.e., including an intercept, and B unknown (i.e., with an intercept and a linear time trend). Clearly, if $d_1 = d_2 = 1$, the model behaves, for $t > 1$, as a random walk vector process if $B_{12} = B_{22} = 0$, and as a random walk with an intercept if $B \neq 0$. In both cases, we observe that there are more non-rejection values compared with the previous case and all them occur for values of d_1 and d_2 close to the unit root. In fact, H_o (34) is rejected for all values of d_1 and d_2 smaller than 0.9 or higher than 1.1, implying that both individual series may contain a unit root. However, the significance of these results may be in large part due to the un-accounted for I(0) autocorrelation in U_t. Thus, we also performed the tests imposing a VAR(1) structure on the disturbances and though the results are not reported across the paper, the conclusions may be summarized as follows: if we do not include regressors, H_o (34) is rejected for all values of d_1 and d_2, and including an intercept and/or a linear trend, the unit root null hypothesis cannot be rejected along with some other (smaller) values for both series. Thus, the results seem to be less nonstationary than in the previous case, but this can be explained by the fact that the parameters in the VAR representation have been obtained using the method of maximum likelihood throughout a quasi-Newton algorithm, and in some cases these parameters can be close to nonstationary. In conclusion, the results based on the multivariate tests support the view that both series contain unit roots which is a preliminary condition if we want to test cointegration between both variables.

7. Conclusions

We have presented different versions of score tests for testing unit roots and other fractionally integrated hypotheses in multivariate systems. They are a natural generalization of the univariate tests of Robinson (1994). The test statistics are expressed in both the time and the frequency domain, using white noise and weakly parametrically autocorrelated disturbances.

Multivariate tests for unit roots have been widely analysed in the literature, especially in the context of cointegration. The test statistics presented in this article do not allow us to test cointegration, however, multivariate unit and fractional root systems appear as particular cases of interest to be tested. In fact, there exists a considerable flexibility in the choice of the null and the alternative hypotheses of the tests, which can entail one or more integer or fractional roots of arbitrary order, for each equation, anywhere on the unit circle in the complex plane. Thus, for example, we can test I(d) and/or quarterly or cyclic I(d) systems of equations.

Results based on Monte Carlo simulations suggest that the test statistics seem to be adequate to test the null of a random walk in a bivariate system. The performance of the tests seem good even for small departures from the null, suggesting that the efficiency property of Robinson's (1994) tests also holds in this multivariate context. The tests were finally applied to the US consumption and income series in order to examine the dynamic behaviour of the series. The results based on the multivariate tests showed that both series may contain unit roots though fractional degrees of integration were also plausible in some cases.

Estimating and testing fractional models in multivariate systems have been semi-parametrically studied among others by Robinson (1995) and Lobato (1999). This article, however, proposes a fully-parametric testing procedure, and given the lack of work in this context, the multivariate tests presented in this paper may be applied to time series data. Extensions of the tests, allowing for a non-diagonal matrix $\Phi(z; \theta)$ should prove relevant to the analysis of fractional and non-fractional cointegration and work in this direction is now in progress.

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Appendix A: Derivation of the score statistic $\hat{S}^t$

The negative of the log-likelihood under (2), (3), (6) and Gaussianity of U_t can be expressed, apart from a constant as

$$L(\theta;\dot{\delta};\dot{\alpha}) = = \frac{T}{2}\log \det K(\dot{\alpha}) + \frac{1}{2}\sum_{t=1}^{T} X_t(\dot{\delta})'\Phi(L;\theta)K(\dot{\alpha})^{-1}\ \Phi(L;\theta)X_t(\dot{\delta}), \qquad \text{(A1)}$$

for any admissible $\dot{\alpha}$ and $\dot{\delta}$, where $U_t(\theta;\dot{\delta}) = \Phi(L;\theta)X_t(\dot{\delta})$ and $X_t(\dot{\delta}) = Y_t - Z_t(\dot{\delta})$.

Starting with the first derivative in (11),

$$\frac{\partial L(\theta;\dot{\delta};\dot{\alpha})}{\partial \theta} = \sum_{t=1}^{T}\left[\frac{\partial \log \rho_1(L;\theta)}{\partial \theta}U_{1t}(\dot{\delta});...;\frac{\partial \log \rho_n(L;\theta)}{\partial \theta}U_{nt}(\dot{\delta})\right]K(\dot{\alpha})^{-1}U_t(\theta;\dot{\delta}),$$

where $U_t(\theta;\dot{\delta}) = (U_{1t}(\theta;\dot{\delta});...;U_{nt}(\theta;\dot{\delta}))'$ and $X_t(\dot{\delta}) = (X_{1t}(\dot{\delta});...;X_{nt}(\dot{\delta}))'$; Evaluating this last expression at θ = 0 we obtain

$$\sum_{t=1}^{T}\left[\varepsilon_{(1)}(L)U_{1t}(\dot{\delta});...;\varepsilon_{(n)}(L)U_{nt}(\dot{\delta})\right]K(\dot{\alpha})^{-1}U_t(\dot{\delta}) \qquad \text{(A2)}$$

where $\varepsilon_{(u)}(L) = \dfrac{\partial \log \rho_u(L;\theta)}{\partial \theta}$ can be expanded as $\sum_{s=1}^{\infty}\psi_s^{(u)}L^s$, in view of (8) and below.

Thus, expression (A2) becomes

$$= \sum_{t=1}^{T}\left[\left(\sum_{s=1}^{\infty}\psi_s^{(1)}U_{1,t-s}(\dot{\delta})\right);...;\left(\sum_{s=1}^{\infty}\psi_s^{(n)}U_{n,t-s}(\dot{\delta})\right)\right]\begin{bmatrix}\sum_{v=1}^{n}\dot{\sigma}^{1v}U_{vt}(\dot{\delta})\\ \cdots\cdots\\ \sum_{v=1}^{n}\dot{\sigma}^{nv}U_{vt}(\dot{\delta})\end{bmatrix} \qquad \text{(A3)}$$

$$= \sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{t=1}^{T-s}U_{ut}(\dot{\delta})\sum_{v=1}^{n}\dot{\sigma}^{uv}U_{v,t+s}(\dot{\delta}) = T\sum_{u=1}^{n}\sum_{v=1}^{n}\dot{\sigma}^{uv}\sum_{s=1}^{T-1}\psi_s^{(u)}C_{uv}(s;\dot{\delta}), \qquad \text{(A4)}$$

where $\dot{\sigma}^{uv}$ is the (u,v)th element of $K(\dot{\alpha})^{-1}$ and $C_{uv}(s;\dot{\delta}) = \frac{1}{T}\sum_{t=1}^{T-s} U_{u,t}(\dot{\delta})U_{v,t+s}(\dot{\delta})$. Calling L_o

$= L(\dot{\eta})_{\theta=0}$, $\frac{\partial L_o}{\partial \dot{\delta}} = \frac{\partial}{\partial \dot{\delta}}\left[\frac{1}{2}\sum_{t=1}^{T} W_t(\dot{\delta})' K(\dot{\alpha})^{-1} W_t(\dot{\delta}) - \sum_{t=1}^{T} W_t(\dot{\delta})' K(\dot{\alpha})^{-1}\Phi(L)Y_t\right] =$

$$\sum_{t=1}^{T}\frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} K(\dot{\alpha})^{-1} W_t(\dot{\delta}) - \sum_{t=1}^{T}\frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} K(\dot{\alpha})^{-1}\Phi(L)Y_t = -\sum_{t=1}^{T}\frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} K(\dot{\alpha})^{-1} U_t(\dot{\delta}). \quad \text{(A5)}$$

From (A1), L_o can also be expressed as

$$\frac{T}{2}\log \det K(\dot{\alpha}) + \frac{1}{2} tr\left[K(\dot{\alpha})^{-1} S(\dot{\delta})\right],$$

where $S(\dot{\delta}) = \sum_{t=1}^{T} U_t(\dot{\delta})U_t(\dot{\delta})'$. Differentiating L_o with respect to $\dot{\alpha}$ leads to:

$$\frac{T}{2} tr\left[K(\dot{\alpha})^{-1}\left(d\, K(\dot{\alpha})\right)\right] - \frac{1}{2} tr\left[K(\dot{\alpha})^{-1}\left(d\, K(\dot{\alpha})\right)K(\dot{\alpha})^{-1} S(\dot{\delta})\right] \quad \text{(A6)}$$

$$= \frac{1}{2} d\, v(K(\dot{\alpha}))' D_m'(K(\dot{\alpha})^{-1} \otimes K(\dot{\alpha})^{-1})\; vec(S(\dot{\delta}) - T\, K(\dot{\alpha})), \quad \text{(A7)}$$

where D_m is the duplication matrix, and using the well known result that tr[ABCD] = (vec A)' (D' ⊗ B) (vec C). Thus, we easily obtain that

$$\frac{\partial L_o}{\partial \dot{\alpha}} = -\frac{1}{2} D_m'(K(\dot{\alpha})^{-1} \otimes K(\dot{\alpha})^{-1})\; vec(S(\dot{\delta}) - T\, K(\dot{\alpha})). \quad \text{(A8)}$$

We next look at the second derivative matrix in (11), and first, we concentrate on the (p x p) matrix $\partial^2 L_o / (\partial \theta \; \partial \theta')$. From the left-hand side in (A4)

$\frac{\partial L_o}{\partial \theta} = \sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{t=1}^{T-s} U_{ut}(\dot{\delta})\sum_{v=1}^{n}\dot{\sigma}^{uv} U_{v,t+s}(\dot{\delta})$, and then, $\frac{\partial^2 L_o}{\partial \theta\, \partial \theta'} =$

$$= \sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{t=1}^{T-s}\left[\left(\sum_{m=1}^{\infty}\psi_m^{(u)}{}' U_{u,t-m}(\dot{\delta})\right)\sum_{v=1}^{n}\dot{\sigma}^{uv} U_{v,t+s}(\dot{\delta}) + U_{ut}(\dot{\delta})\sum_{v=1}^{n}\dot{\sigma}^{uv}\left(\sum_{m=1}^{\infty}\psi_m^{(v)}{}' U_{v,t+s-m}(\dot{\delta})\right)\right].$$

In order to form (11), we need to take the expectation of this last expression. (Note that it is evaluated at θ = 0, i.e. under the null (5)). This expectation is zero for the first summand, given the uncorrelatedness in U_t and since it involves terms of the form $U_{u,t-m}$ and $U_{v,t+s}$, for m,s > 0. The expectation for the second summand is

$$\sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{v=1}^{n}\dot{\sigma}^{uv}\psi_s^{(v)}{}'\sum_{t=1}^{T-s}E(U_{ut}(\dot{\delta})U_{vt}(\dot{\delta})) \;=\; T\sum_{s=1}^{T-1}\left(1-\frac{s}{T}\right)\sum_{u=1}^{n}\sum_{v=1}^{n}\dot{\sigma}^{uv}\dot{\sigma}_{uv}\psi_s^{(u)}\psi_s^{(v)}{}'.$$

Again from (A4), we have that

$$\frac{\partial L_o}{\partial\theta} = \sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{t=1}^{T-s}\left(\rho_u(L)Y_{ut} - W_{ut}(\dot{\delta})\right)\sum_{v=1}^{n}\dot{\sigma}^{uv}\left(\rho_v(L)Y_{v,t+s} - W_{v,t+s}(\dot{\delta})\right)$$

and from this expression we obtain that $\frac{\partial^2 L_o}{\partial\theta\partial\dot{\delta}'} =$

$$\frac{\partial}{\partial\dot{\delta}'}\left[\sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{t=1}^{T-s}\left(W_{ut}\sum_{v=1}^{n}\dot{\sigma}^{uv}W_{v,t+s} - \rho_u(L)Y_{ut}\sum_{v=1}^{n}\dot{\sigma}^{uv}W_{v,t+s} - W_{ut}\sum_{v=1}^{n}\dot{\sigma}^{uv}\rho_v(L)Y_{v,t+s}\right)\right] =$$

$$= \sum_{u=1}^{n}\sum_{s=1}^{T-1}\psi_s^{(u)}\sum_{v=1}^{n}\dot{\sigma}^{uv}\sum_{t=1}^{T-s}\left[\frac{\partial W_{u,t}(\dot{\delta})}{\partial\dot{\delta}'}U_{v,t+s}(\dot{\delta}) + U_{u,t}(\dot{\delta})\frac{\partial W_{v,t+s}(\dot{\delta})}{\partial\dot{\delta}'}\right]. \qquad (A9)$$

For the derivation of $\frac{\partial^2 L_o}{\partial\theta\,\partial\dot{\alpha}'}$, defining $P_t(\dot{\delta}) = [P_{1t}(\dot{\delta});\ldots;P_{nt}(\dot{\delta})]$ the (p x n) matrix appearing in (A3), then $\frac{\partial L_o}{\partial\theta} = \sum_{t=1}^{T}P_t(\dot{\delta})K(\dot{\alpha})^{-1}U_t(\dot{\delta})$, and differentiating this expression with respect to $\dot{\alpha}$ leads to:

$$\frac{\partial^2 L_o}{\partial\theta\,\partial\dot{\alpha}} = -\sum_{t=1}^{T}\left(U_t(\dot{\delta})\otimes P_t(\dot{\delta})\right)\left(\dot{K}^{-1}(\dot{\alpha})\otimes\dot{K}^{-1}(\dot{\alpha})\right)D_m. \qquad (A10)$$

Finally, in order to complete the Hessian in (11), we still have to calculate some second derivatives with respect to $\dot{\delta}$ and $\dot{\alpha}$. From (A5)

$$\frac{\partial^2 L_o}{\partial\dot{\delta}\,\partial\dot{\delta}'} = \sum_{t=1}^{T}\left(\frac{\partial W_t(\dot{\delta})}{\partial\dot{\delta}}K(\dot{\alpha})^{-1}\frac{\partial W_t(\dot{\delta})}{\partial\dot{\delta}'} - \left(U_t(\dot{\delta})'K(\dot{\alpha})^{-1}\otimes I_k\right)\frac{\partial vec}{\partial\dot{\delta}'}\left(\frac{\partial W_t(\dot{\delta})}{\partial\dot{\delta}}\right)\right). \qquad (A11)$$

We next consider $\frac{\partial^2 L_o}{\partial \dot{\delta}\, \partial \dot{\alpha}'}$, and since $\frac{\partial L_o}{\partial \dot{\delta}} = -\sum_{t=1}^{T} \frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} K(\dot{\alpha})^{-1} U_t(\dot{\delta})$, differentiating this last expression with respect to $\dot{\alpha}$, we obtain

$$\sum_{t=1}^{T} \frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} K(\dot{\alpha})^{-1} d(K(\dot{\alpha}))K(\dot{\alpha})^{-1} U_t(\dot{\delta}) = \sum_{t=1}^{T} \left(U_t(\dot{\delta}) \otimes \frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} \right) vec\left[K(\dot{\alpha})^{-1} d(K(\dot{\alpha}))K(\dot{\alpha})^{-1} \right]$$

$$= \sum_{t=1}^{T} \left(U_t(\dot{\delta}) \otimes \frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} \right) \left(K(\dot{\alpha})^{-1} \otimes K(\dot{\alpha})^{-1} \right) D_m \, d\, v(K(\dot{\alpha})).$$ Thus,

$$\frac{\partial^2 L_o}{\partial \dot{\delta}\, \partial \dot{\alpha}'} = \sum_{t=1}^{T} \left(U_t(\dot{\delta})' \otimes \frac{\partial W_t(\dot{\delta})}{\partial \dot{\delta}} \right) \left(K(\dot{\alpha})^{-1} \otimes K(\dot{\alpha})^{-1} \right) D_m. \qquad \text{(A12)}$$

Finally, we look at $\frac{\partial^2 L_o}{\partial \dot{\alpha}\, \partial \dot{\alpha}'}$. Differentiating (A6) with respect to $\dot{\alpha}$,

$$-\frac{T}{2} tr\left[\dot{K}^{-1}(d\dot{K})\, \dot{K}^{-1}(d\dot{K}) \right] + \frac{1}{2} tr\left[\dot{K}^{-1}(d\dot{K})\, \dot{K}^{-1}(d\dot{K})\, \dot{K}^{-1} S(\dot{\delta}) \right] + \frac{1}{2} tr\left[\dot{K}^{-1}(d\dot{K})\, \dot{K}^{-1}(d\dot{K})\, \dot{K}^{-1} S(\dot{\delta}) \right]$$

obtaining that

$$\frac{\partial^2 L_o}{\partial \alpha\, \partial \alpha'} = -\frac{T}{2} D_m' \left(\dot{K}^{-1} \otimes \dot{K}^{-1} \right) D_m + D_m' \left(\dot{K}^{-1} S(\dot{\delta}) \dot{K}^{-1} \otimes \dot{K}^{-1} \right) D_m.$$

We can get now consistent and efficient estimates of δ and α by equating (A5) and (A8) to zero; however, for practical purposes and in order to simplify the computations, we can take any $T^{1/2}$-consistent estimates of δ and α. We will assume that $\hat{\delta}$ is a consistent estimate of δ and we will take $\hat{K} = K(\hat{\alpha}) = T^{-1} S(\hat{\delta})$. It follows from previous pages that

$$\frac{\partial L(0; \hat{\delta}; \hat{\alpha})}{\partial \theta} = \sum_{u=1}^{n} \sum_{s=1}^{T-1} \psi_s^{(u)} \sum_{t=1}^{T-s} \hat{U}_{u,t}(\delta) \sum_{v=1}^{n} \hat{\sigma}^{uv} \hat{U}_{v,t+s}(\delta) = T \sum_{u=1}^{n} \sum_{s=1}^{T-1} \sum_{v=1}^{n} \psi_s^{(u)}\, \hat{\sigma}^{uv} C_{uv}(s; \hat{\delta}).$$

Also, $E\left(\frac{\partial^2 L(0;\hat{\delta}\,\hat{\alpha})}{\partial\theta\,\partial\theta'}\right) = T\sum_{u=1}^{n}\sum_{v=1}^{n}\hat{\sigma}^{uv}\hat{\sigma}_{uv}\sum_{s=1}^{T-s}\left(1-\frac{s}{T}\right)\psi_s^{(u)}\psi_s^{(v)}{'} = T\hat{A}^t$, and the asymptotic expectation matrix in (11) multiplied by 1/T will take the form

$$\begin{pmatrix} \overline{A} & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & \frac{1}{2}D_m{'}(K^{-1}\otimes K^{-1})D_m \end{pmatrix} \tag{A13}$$

where $\overline{A} = \sum_{u=1}^{n}\sum_{v=1}^{n}\sigma^{uv}\sigma_{uv}\sum_{s=1}^{\infty}\psi_s^{(u)}\psi_s^{(v)}{'}$ and $W = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\partial W_t(\delta)}{\partial\delta}K^{-1}\frac{\partial W_t(\delta)}{\partial\delta'}\right)$ is a positive definite matrix by assumption. (Note that the block diagonality in (A13) follows from expressions (A9), (A10) and (A12), given that $\hat{\sigma}^{uv}$ consistently estimates σ^{uv} and $\hat{U}_t(\delta)$ has zero expectation).

Appendix B: Derivation of the score statistic $\widetilde{S}$

For the derivation of the score test statistic in the context of weakly autocorrelated U_t, we assume that k and K in (20) are parameterized separately, so $\dot{\tau}$ is taken to specify k and $\dot{\alpha}$ to specify K. Thus, the spectral density matrix of $U_t(\theta;\dot{\delta})$ for any admissible $\dot{\delta}$ and $\dot{\tau}$ is

$$f(\lambda;\dot{\alpha};\dot{\tau}) \;=\; \frac{1}{2\pi}k(\lambda;\dot{\tau})\,K(\dot{\alpha})k(\lambda;\dot{\tau})^{*} \qquad \text{(B1)}$$

where $k(\lambda;\dot{\tau}) = \sum_{j=0}^{\infty} A(j;\dot{\tau})e^{i\lambda j}$. It is also assumed that $A(0;\dot{\tau}) = I_n$ (the n-rowed identity matrix) for any $\dot{\tau}$ in the Euclidean space R^q, and that $f(\lambda;\dot{\alpha};\dot{\tau})$ is a finite, positive matrix with eigenvalues bounded and bounded away from zero at any frequency on the neighbourhood N^* of τ and M^* of α. Also, we assume that each element of $\hat{f}(\lambda;\dot{\tau})$, $\hat{f}_{uv}(\lambda;\dot{\tau})$, as defined below (B4), must be continuous in (λ, $\dot{\tau}$) for $\dot{\tau} \in N^*$ and have first and second derivatives with respect to τ continuous in (λ, $\dot{\tau}$) for $\dot{\tau} \in N^*$.

Taking η = (θ'; α'; δ'; τ')', the negative of the log-likelihood based on Gaussianity of U_t can be expressed as

$$l(\dot{\eta}) \;=\; \frac{1}{2}\log\,\det\, J(\dot{\alpha};\dot{\tau}) \;+\; \frac{1}{2}U(\theta;\dot{\delta})'J^{-1}(\dot{\alpha};\dot{\tau})U(\theta;\dot{\delta}), \qquad \text{(B2)}$$

where $U(\theta;\dot{\delta}) = (U_1(\theta;\dot{\delta});...;U_T(\theta;\dot{\delta}))'$, and $J(\dot{\alpha};\dot{\tau})$ is a (nT x nT) matrix with $J_{s-t}(\dot{\alpha};\dot{\tau}) \;=\; \int_{-\pi}^{\pi} e^{i(s-t)\lambda}\, f(\lambda;\dot{\alpha};\dot{\tau})\,d\lambda$ in the (t,s) block of n^2 elements, for any admissible $\dot{\alpha}$,

$\dot{\delta}$ and $\dot{\tau}$. However, given the computational difficulty of this expression, especially when n and T are large, under suitable conditions, (B2) can be approximated by

$$L(\theta;\dot{\alpha};\dot{\delta};\dot{\tau}) \;=\; \frac{T}{2}\log\,\det\, f(\lambda_r;\dot{\alpha};\dot{\tau}) \;+\; \frac{1}{2}\sum_r^{*} tr\left[f^{-1}(\lambda_r;\dot{\alpha};\dot{\tau})\, I_U(\lambda_r;\theta;\dot{\delta})\right], \qquad \text{(B3)}$$

where $I_U(\lambda_r;\theta;\dot{\delta})$ is the periodogram of $U_t(\theta;\dot{\delta})$ evaluated at frequencies $\lambda_r = 2\pi r/T$ and the sum on * is as described in Section 2.

Calling $\hat{\delta}$ any $T^{1/2}$-consistent estimate of δ, and $\hat{\alpha}$ as defined in Appendix A, we can concentrate both out and consider

$$\hat{L}(\theta;\dot{\tau}) \;=\; L(\theta;\hat{\alpha};\hat{\delta};\dot{\tau}) \;=\; \frac{T}{2}\log\,\det \hat{f}(\lambda_r;\dot{\tau}) \;+\; \frac{1}{2}\sum_r^{*} tr\left[\hat{f}^{-1}(\lambda_r;\dot{\tau})\, \hat{I}_U(\lambda_r;\theta)\right], \qquad \text{(B4)}$$

where $\hat{f}(\lambda_r;\dot{\tau}) = \frac{1}{2\pi} k(\lambda_r;\dot{\tau})\, K(\hat{\alpha})\, k(\lambda_r;\dot{\tau})^{*}$, and $\hat{I}_U(\lambda_r;\theta) = \frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T} U_t(\theta;\hat{\delta})\, e^{i\lambda_r t}$.

Thus, we can express a score test statistic as:

$$\frac{\partial \hat{L}(\theta;\dot{\tau})}{\partial \theta'}\left[E\left(\frac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\theta\,\partial\theta'}\right) - E\left(\frac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\theta\,\partial\dot{\tau}'}\right)\left(E\left(\frac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\dot{\tau}\,\partial\dot{\tau}'}\right)\right)^{-1} E\left(\frac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\dot{\tau}\,\partial\theta'}\right)\right]^{-1} \frac{\partial \hat{L}(\theta;\dot{\tau})}{\partial \theta}\Bigg|_{\substack{\theta=0\\ \tau=\tilde{\tau}}} \qquad \text{(B5)}$$

where the expectation is taken under the null hypothesis (5) prior to substitution of $\tilde{\tau}$, where $\tilde{\tau}$ can be any consistent estimate of τ under (5).

We start with $\partial \hat{L}(\theta;\dot{\tau})/\partial\theta$, and from (B4), we see that it is

$$= \frac{1}{2}\sum_r^{*}\sum_{u=1}^{n}\sum_{v=1}^{n} \frac{\partial \hat{I}_{uv}(\lambda_r;\theta)}{\partial\theta}\, \hat{f}^{vu}(\lambda_r;\dot{\tau}), \qquad \text{(B6)}$$

where $\hat{I}_{uv}(\lambda_r;\theta)$ is the (u,v)th element of $\hat{I}_U(\lambda_r;\theta)$, and $\hat{f}^{uv}(\lambda_r;\dot{\tau})$ is the (u,v)th element of $\hat{f}^{-1}(\lambda_r;\dot{\tau})$. We first concentrate on

$$\frac{\partial \hat{I}_{uv}(\lambda_r;\theta)}{\partial\theta}\Big|_{\theta=0} \;=\; \frac{\partial}{\partial\theta}\left(\frac{1}{2\pi T}\sum_{t=1}^{T}\sum_{s=1}^{T} U_{u,t}(\theta;\hat{\delta})\, U_{v,t}(\theta;\hat{\delta})\, e^{i(t-s)\lambda_r}\right)_{|\,\theta=0} \;=$$

$$\sum_{m=1}^{T-1}\psi_m^{(u)}\, e^{i\lambda_r m}\,\frac{1}{2\pi T}\sum_{t=1}^{T-m}\sum_{s=1}^{T}U_{ut}(\hat{\delta})U_{vs}(\hat{\delta})e^{i\lambda_r(t-s)} + \sum_{m=1}^{T-1}\psi_m^{(v)}\, e^{-i\lambda_r m}\,\frac{1}{2\pi T}\sum_{t=1}^{T}\sum_{s=1}^{T-m}U_{ut}(\hat{\delta})U_{vs}(\hat{\delta})\, e^{i\lambda_r(t-s)}$$

and, under suitable conditions, (with m = 1,2,…,M < T-1, for sufficiently large M), this expression becomes asymptotically: $\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)I_{uv}(\lambda_r;\hat{\delta}).$ (B7)

Substituting (B7) in (B6), we obtain that $\frac{\partial \hat{L}(\theta;\dot{\tau})}{\partial\theta}\Big|_{\theta=0}$ is asymptotically

$$\frac{1}{2}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)I_{uv}(\lambda_r;\hat{\delta})\,\hat{f}^{vu}(\lambda_r;\dot{\tau}). \qquad \text{(B8)}$$

We next examine the second derivative matrices in (B5). We start with

$$\frac{\partial^2\,\hat{L}(\theta;\dot{\tau})}{\partial\theta\,\partial\theta'} = \frac{1}{2}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)\frac{\partial\hat{I}_{uv}(\lambda_r;\theta)}{\partial\theta'}\,\hat{f}^{vu}(\lambda_r;\dot{\tau})$$

and using again (B7), this last expression evaluated at θ = 0, becomes for large T

$$\frac{1}{2}\sum_{r}^{*}\sum_{t=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)\left(\varepsilon_{(u)}(\lambda_r)' + \bar{\varepsilon}_{(v)}(\lambda_r)'\right)I_{uv}(\lambda_r;\hat{\delta})\;\hat{f}^{vu}(\lambda_r;\dot{\tau}),$$

whose asymptotic expectation is

$$\frac{1}{2}\sum_{r}^{*}\sum_{t=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)\left(\varepsilon_{(u)}(\lambda_r)' + \bar{\varepsilon}_{(v)}(\lambda_r)'\right)\hat{f}_{uv}(\lambda_r;\dot{\tau})\;\hat{f}^{vu}(\lambda_r;\dot{\tau}),$$

given that, heuristically, if $f(\lambda;\tau)$ is continuous in λ, $E(I_{uv}(\lambda)) \to_{T\to\infty} f_{uv}(\lambda;\tau)$, for fixed λ. (See Brillinger, 1981). We can write this last expression as

$$\frac{1}{2}\sum_{r}^{*}\sum_{u=1}^{n}\varepsilon_{(u)}(\lambda_r)\,\varepsilon_{(u)}(\lambda_r)'\sum_{v=1}^{n}\hat{f}_{uv}(\lambda_r;\tilde{\tau})\hat{f}^{vu}(\lambda_r;\tilde{\tau}) + \frac{1}{2}\sum_{r}^{*}\sum_{v=1}^{n}\bar{\varepsilon}_{(v)}(\lambda_r)\,\bar{\varepsilon}_{(v)}(\lambda_r)'\sum_{u=1}^{n}\hat{f}_{uv}(\lambda_r;\tilde{\tau})\hat{f}^{vu}(\lambda_r;\tilde{\tau})$$
$$+\frac{1}{2}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n}\left(\varepsilon_{(u)}(\lambda_r)\bar{\varepsilon}_{(v)}(\lambda_r)' + \bar{\varepsilon}_{(v)}(\lambda_r)\varepsilon_{(u)}(\lambda_r)'\right)\hat{f}_{uv}(\lambda_r;\dot{\tau})\hat{f}^{vu}(\lambda_r;\dot{\tau}), \qquad \text{(B9)}$$

which first two summands will be approximately zero, noting that

$$\sum_{v=1}^{n} \hat{f}_{uv}(\lambda_r;\dot{\tau})\hat{f}^{vu}(\lambda_r;\dot{\tau}) = \sum_{u=1}^{n} \hat{f}_{uv}(\lambda_r;\dot{\tau})\hat{f}^{vu}(\lambda_r;\dot{\tau}) = 1, \text{ and}$$

$$\sum_{r}^{*} \varepsilon_{(u)}(\lambda_r)\,\varepsilon_{(u)}(\lambda_r)' = \sum_{r}^{*} \bar{\varepsilon}_{(v)}(\lambda_r)\,\bar{\varepsilon}_{(v)}(\lambda_r)' = 0, \text{ for all u,v = 1,2,…,n.} \qquad \text{(B10)}$$

We next look at the (p x q) matrix $\dfrac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\theta\,\partial\dot{\tau}'}$ in (B5) which, evaluated at θ = 0, is

$$\frac{\partial}{\partial\dot{\tau}'}\left(\frac{1}{2}\sum_{r}^{*}\sum_{\substack{u=1\\v=1}}^{n} \left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right) I_{uv}(\lambda_r;\hat{\delta})\,\hat{f}^{vu}(\lambda_r;\dot{\tau}) \right) = \frac{1}{2}\sum_{r}^{*}\sum_{\substack{u=1\\v=1}}^{n} \left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right) I_{uv}(\lambda_r;\hat{\delta})\frac{\partial \hat{f}^{vu}(\lambda_r;\dot{\tau})}{\partial\dot{\tau}'},$$

whose expectation for large T is

$$\frac{1}{2}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n} \left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)\hat{f}_{uv}(\lambda_r;\dot{\tau})\frac{\partial \hat{f}^{vu}(\lambda_r;\dot{\tau})}{\partial\dot{\tau}'}. \qquad \text{(B11)}$$

This last expression can also be described by the derivatives of $\hat{f}$ with respect to $\dot{\tau}$. Thus, (B11) can be expressed as

$$-\frac{1}{2}\sum_{r}^{*}\sum_{u=1}^{n}\sum_{v=1}^{n} \left(\varepsilon_{(u)}(\lambda_r) + \bar{\varepsilon}_{(v)}(\lambda_r)\right)\hat{f}^{vu}(\lambda_r;\dot{\tau})\frac{\partial \hat{f}_{uv}(\lambda_r;\dot{\tau})}{\partial\dot{\tau}'}. \qquad \text{(B12)}$$

Finally, we look at the (q x q) matrix $\dfrac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\dot{\tau}\,\partial\dot{\tau}'}$. The u$^{\text{th}}$ element of $\dfrac{\partial \hat{L}(\theta;\dot{\tau})}{\partial\dot{\tau}}$ is

$$\frac{1}{2}\sum_{r}^{*} tr\left(\hat{f}^{-1}(\lambda_r;\dot{\tau})\frac{\partial \hat{f}(\lambda_r;\dot{\tau})}{\partial\dot{\tau}_u} - \hat{f}^{-1}(\lambda_r;\dot{\tau})\frac{\partial \hat{f}(\lambda_r;\dot{\tau})}{\partial\dot{\tau}_u}\hat{f}^{-1}(\lambda_r;\dot{\tau})\hat{I}_U(\lambda_r;\theta) \right).$$

Then, $\dfrac{\partial^2 \hat{L}(\theta;\dot{\tau})}{\partial\dot{\tau}_u\,\partial\dot{\tau}_v}$, evaluated at θ = 0 becomes

$$\frac{1}{2}\sum_{r}^{*} tr\left(-\hat{f}^{-1}\frac{\partial\hat{f}}{\partial\dot{\tau}_v}\hat{f}^{-1}\frac{\partial\hat{f}}{\partial\dot{\tau}_u} + \hat{f}^{-1}\frac{\partial^2\hat{f}}{\partial\dot{\tau}_u\partial\dot{\tau}_v} + \hat{f}^{-1}\frac{\partial\hat{f}}{\partial\dot{\tau}_v}^{-1}\frac{\partial\hat{f}}{\partial\dot{\tau}_u}\hat{f}^{-1}\hat{I}_U - \hat{f}^{-1}\frac{\partial^2\hat{f}}{\partial\dot{\tau}_u\partial\dot{\tau}_v}\hat{f}^{-1}\hat{I}_U - \hat{f}^{-1}\frac{\partial\hat{f}}{\partial\dot{\tau}_v}\hat{f}^{-1}\frac{\partial\hat{f}}{\partial\dot{\tau}_v}\hat{f}^{-1}\hat{I}_U \right),$$

whose asymptotic expectation is

$$\frac{1}{2}\sum_{r}^{*} tr\left(\hat{f}^{-1}(\lambda_r;\dot{\tau})\frac{\partial \hat{f}(\lambda_r;\dot{\tau})}{\partial \dot{\tau}_u}\hat{f}^{-1}(\lambda_r;\dot{\tau})\frac{\partial \hat{f}(\lambda_r;\dot{\tau})}{\partial \dot{\tau}_v} \right). \tag{B13}$$

Substituting (B8), (B9), (B11) and (B13) in (B5), at $\dot{\tau} = \tilde{\tau}$, we form (21).

TABLE 1

Rejection frequencies of $\hat{S}^{t_2}$ in (30) with $\Sigma = I$

True model: $\theta_1 = \theta_2 = 0$; $\alpha = 5\%$ No. of replications: 5,000

Table 1a): T = 50

θ_1 / θ_2	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
-0.8	.999	.998	.981	.889	.744	.811	.947	.993	.999
-0.6	.997	.986	.918	.708	.462	.601	.879	.984	.998
-0.4	.980	.919	.726	.400	.181	.358	.778	.962	.996
-0.2	.888	.700	.394	.132	.039	.209	.660	.935	.990
0	.741	.457	.170	.037	**.012**	.147	.585	.910	.987
0.2	.805	.591	.350	.205	.137	.256	.646	.923	.989
0.4	.944	.873	.769	.659	.582	.648	.846	.968	.996
0.6	.993	.977	.959	.927	.904	.920	.967	.992	.998
0.8	1.000	.999	.996	.991	.987	.988	.997	.999	1.000

Table 1b): T = 100

θ_1 / θ_2	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
-0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.6	1.000	1.000	1.000	.998	.989	.996	1.000	1.000	1.000
-0.4	1.000	1.000	.998	.958	.803	.925	.998	1.000	1.000
-0.2	1.000	.998	.958	.618	.221	.632	.976	1.000	1.000
0	1.000	.987	.798	.206	**.020**	.426	.945	.999	1.000
0.2	1.000	.996	.922	.637	.425	.693	.976	.999	1.000
0.4	1.000	1.000	.997	.975	.949	.975	.997	1.000	1.000
0.6	1.000	1.000	1.000	.999	.999	1.000	1.000	1.000	1.000
0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1c):				T = 200					
θ_1 / θ_2	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
-0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.4	1.000	1.000	1.000	1.000	.999	1.000	1.000	1.000	1.000
-0.2	1.000	1.000	1.000	.997	.680	.966	1.000	1.000	1.000
0	1.000	1.000	.999	.671	**.032**	.793	.999	1.000	1.000
0.2	1.000	1.000	.999	.969	.808	.972	1.000	1.000	1.000
0.4	1.000	1.000	1.000	.999	.999	.999	1.000	1.000	1.000
0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

In bold, the size of the tests. The nominal size is 0.05.

TABLE 2									
Rejection frequencies of $\hat{S}^{t_2}$ in (30) with $\sum = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$									
True model: $\theta_1 = \theta_2 = 0$;				α = 5 %			No. of replications: 5,000		
Table 2a):				T = 50					
θ_1 / θ_2	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
-0.8	.999	.998	.990	.979	.986	.997	.999	1.000	1.000
-0.6	.997	.987	.932	.866	.921	.986	.998	.999	1.000
-0.4	.988	.930	.743	.503	.625	.920	.992	.999	1.000
-0.2	.975	.861	.500	.152	.144	.688	.964	.996	.999
0	.986	.913	.618	.149	**.014**	.327	.882	.987	.999
0.2	.996	.984	.916	.685	.340	.256	.766	.978	.997
0.4	.999	.998	.992	.960	.879	.764	.846	.976	.997
0.6	1.000	1.000	.999	.998	.991	.979	.977	.992	.999
0.8	1.000	1.000	.999	.999	.999	.999	.999	.999	1.000
Table 2b):				T = 100					
θ_1 / θ_2	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
-0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.4	1.000	1.000	.999	.982	.993	1.000	1.000	1.000	1.000
-0.2	1.000	.999	.983	.630	.571	.989	1.000	1.000	1.000
0	1.000	1.000	.992	.594	**.021**	.750	.998	1.000	1.000
0.2	1.000	1.000	1.000	.985	.754	.689	.988	1.000	1.000
0.4	1.000	1.000	1.000	1.000	.999	.989	.997	1.000	1.000
0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Table 2c):				T = 200					

θ_1 / θ_2	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
-0.8	1.000	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.6	1.000	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.4	1.000	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.2	1.000	1.00	1.000	.979	.970	1.000	1.000	1.000	1.000
0	1.000	1.00	1.000	.968	**.032**	.979	1.000	1.000	1.000
0.2	1.000	1.00	1.000	1.000	.980	.972	1.000	1.000	1.000
0.4	1.000	1.00	1.000	.999	1.000	.999	1.000	1.000	1.000
0.6	1.000	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.8	1.000	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000

In bold, the size of the tests. The nominal size is 0.05.

TABLE 3

Table 3a: Empirical sizes of $\hat{S}^{f_2}$ in (31) with $\Sigma = I_2$

True model: $\theta_1 = \theta_2 = 0$ No. of replications: 5000

T / α	10%	5%	2.5%	1%
50	0.028	0.012	0.001	0.000
100	0.058	0.019	0.010	0.006
200	0.074	0.038	0.020	0.008

Table 3b: Empirical sizes of $\hat{S}^{f_2}$ in (31) with Σ = [(1, 1)'; (1, 2)']

True model: $\theta_1 = \theta_2 = 0$ No. of replications: 5000

T / α	10%	5%	2.5%	1%
50	0.036	0.012	0.002	0.000
100	0.057	0.021	0.008	0.005
200	0.066	0.035	0.017	0.006

TABLE 4

Table 4a: Empirical sizes of $\widetilde{S}^{2}$ in (32) with a VAR(1) structure on

True model: $\theta_1 = \theta_2 = 0$ No. of replications: 5000

T / α	10%	5%	2.5%	1%
50	0.134	0.074	0.040	0.017
100	0.123	0.069	0.035	0.014

200	0.104	0.060	0.031	0.012

Table 4b: Empirical sizes of $\tilde{S}^2$ in (32) with a VMA(1) structure on

True model: $\theta_1 = \theta_2 = 0$				No. of replications: 5000
T / α	10%	5%	2.5%	1%
50	0.207	0.154	0.127	0.097
100	0.137	0.090	0.054	0.045
200	0.131	0.062	0.038	0.023

TABLE 5

Multivariate score tests in the frequency domain ($\hat{S}^{t_2}$ in (30)) with white noise U_t

Table 5a): With no regressors

d_1 / d_2	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
0.6	122.55	172.91	215.06	221.57	212.69	198.34	182.67	167.51	153.64
0.7	104.72	33.79	111.52	157.70	166.76	160.97	150.09	138.00	126.94
0.8	184.68	61.22	7.02	72.15	112.35	122.14	119.24	111.84	103.25
0.9	217.32	143.00	40.55	**3.84**	49.58	80.36	89.56	88.83	84.32
1.0	217.96	168.65	104.29	26.61	7.64	38.33	60.21	68.05	68.64
1.1	206.37	167.29	124.77	72.11	18.89	13.15	33.84	48.84	55.11
1.2	190.92	156.84	124.29	89.42	49.42	16.82	18.77	32.97	43.16
1.3	175.06	143.98	116.71	90.87	64.12	36.09	18.60	23.98	33.90
1.4	160.21	131.23	107.28	86.50	67.29	48.12	30.03	22.35	28.64

Table 5b): With an intercept

d_1 / d_2	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
0.6	157.37	95.36	120.74	157.40	179.09	188.52	191.90	192.86	193.02
0.7	203.96	68.94	37.29	53.74	72.31	83.64	89.71	92.96	94.82
0.8	247.10	88.92	16.75	8.94	17.74	26.10	31.92	35.89	38.69
0.9	270.05	112.12	23.24	**0.85**	**2.11**	6.99	11.33	14.81	17.60
1.0	278.38	126.19	33.18	**4.30**	**1.70**	**4.59**	7.85	10.75	13.25
1.1	279.82	132.92	40.49	9.32	**4.98**	6.96	9.71	12.29	14.59
1.2	279.00	135.81	45.22	13.64	8.60	10.19	12.75	15.20	17.42
1.3	277.89	137.00	48.29	17.05	11.81	13.27	15.77	18.22	20.43
1.4	277.11	137.53	50.38	19.73	14.53	15.97	18.48	20.96	23.21

Table 5c): With a linear time trend

d_1 / d_2	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4

0.6	69.22	39.25	44.85	59.61	72.38	80.83	85.72	88.27	89.49
0.7	80.97	27.07	14.93	20.92	30.72	39.06	44.96	48.79	51.18
0.8	95.91	31.25	6.44	**3.75**	8.92	15.27	20.64	24.67	27.54
0.9	106.91	40.06	8.77	**0.23**	**1.55**	**5.72**	10.01	13.62	16.44
1.0	113.30	48.03	14.46	**2.80**	**1.70**	**4.32**	7.69	10.80	13.39
1.1	116.36	53.75	20.08	7.11	**4.73**	6.45	9.26	12.03	14.43
1.2	117.48	57.43	24.56	11.30	8.37	9.65	12.18	14.80	17.12
1.3	117.76	59.67	27.85	14.79	11.72	12.83	15.26	17.85	20.17
1.4	117.54	61.04	30.21	17.55	14.54	15.62	18.04	20.66	23.04

In bold, the non-rejection values of the null hypothesis (34) at the 95% significance level.